Subtraction and Negative Numbers:

Examining the Problem Size Effect in Mixed Formats[1]

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Abstract

Every simple subtraction problem can formally be stated that for any integers a and b, a – b = a + (-b). Therefore a simple subtraction problem can be stated equivalently by "adding the opposite." Although, the problems are theoretically the same, the addition of negative numbers is assumed to be difficult to conceptualize. In the following experiment the authors have compared response times and error rates for corresponding additive and subtractive forms of single-digit arithmetic problems. The authors found that participants solved problems of the form a – b significantly faster than problems of the form a + (-b). Furthermore, there was no evident problem size effect and thus an interference theory was suggested.

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Mathematical cognition has become an area of great interest in the past 30 years. The reason for such interest and, furthermore, research may be attributed to the influence of arithmetic knowledge in our daily lives. Knowledge of arithmetic and the number system is acquired during early education and forms the foundation for higher level mathematics, problem solving, and mathematical reasoning (Seyler, Kirk, & Ashcraft, 2003).

Unequivocally, the most important variable in the area of mental arithmetic is problem size. Within the area of mental arithmetic, considerable attention has been devoted to the problem size effect in the study of simple addition and multiplication (Seyler et al., 2003; LeFevre, Sadesky, & Bisanz, 1996; Penner-Wilger, Leth-Steensen, & LeFevre, 2002). Problem size is the phenomenon whereby problems composed of larger numbers are solved more slowly and less accurately than problems composed of smaller numbers (LeFevre et al., 1996). For example, response time and errors will be larger for the problem 9 + 7 than for 5 + 3 (Seyler et al., 2003). Furthermore, the shape and slope of the problem size effect have been used as evidence for a variety of important explanations of mental arithmetic performance, including counting (Shrager & Siegler, 1998) versus retrieval (Ashcraft & Battaglia, 1978) performance in addition and encoding (Blankenberger, 2001) versus accessibility theories (LeFevre et al., 1996) with regards to simple arithmetic. Therefore, any complete theory of mental arithmetic should account for the problem size effect (LeFevre et al., 1996).

Interestingly, a limited amount of research has been conducted on elementary subtraction, the basic facts of subtraction taught universally in second grade. Moreover few studies on mathematical cognition contain information on the problem size effect in subtraction (Seyler et al., 2003). Simple subtraction involves problems of the form $m - s = r$, whereby m denotes the minuend, *s* denotes the subtrahend, and *r* denotes the remainder or difference. A small problem consists of minuends in the range of 0-9. Large problems are those that consist of minuends with a range of 10-18. The basic facts of subtraction are the inverses of addition facts. Therefore using the same labels, the inverse of $m - s = r$ would be $s + r = m$, augend (*s*, the first operand) plus addend (*r*, the second operand) equal sum *m* (Seyler et al., 2003).

Seyler et al. (2003) demonstrated a strong relationship between response times and the minuend in simple subtraction. In particular, a pronounced jump occurred in response times beginning with a minuend of 11, rather than a continuous increase from previous minuends. As such, the researchers suggested that longer response times of large subtraction problems involve a mixture of processing operations. Furthermore, Seyler et al. (2003) concluded that there is a larger problem size effect in subtraction than seen in addition or multiplication.

Additionally, limited research has been conducted on the relationship between negative numbers and positive numbers in simple arithmetic. There have been some studies on the mental representation of negative numbers (Shaki & Petrusic, in press) and there is research on the conceptual changes that occur when students deal with negative numbers in elementary algebraic operations (Vlassis, 2004). Also, increasing reference to the SNARC effect (i.e., Spatial Numerical Association of Response Codes) has been made in studies of negative numbers. The SNARC effect demonstrates the spatial representation of positive and negative numbers by examining parity judgments (i.e., the relationship of oddness or evenness between two numbers) (Dehaene, Bossini, & Giraux, 1993). Although the SNARC effect is a useful examination of the mental representation of the number line, it is not as useful for predicting the relationship between negative numbers and problem solving.

More specifically, few researchers have compared the basic processing characteristics of negative addition and simple subtraction among adults, and none have discussed the results with a great level of detail. Negative addition problems are related to subtraction and require subtraction for their solution (Peterson & Aller, 1971). Furthermore, the study of negative addition is important as it is taught in school curriculums (Vlassis, 2004). Given the importance of the relationship between negative addition and simple subtraction and the absence of detailed information in the literature, our main purpose was to conduct and present foundation work on the problem size effect in negative addition and simple subtraction.

Two published studies on different arithmetic operations deserve mention here. Dixon, Deets, and Bangert (2001) tested whether or not people represent analogous principles for each arithmetic operation (i.e., addition, subtraction, multiplication, and division). Participants rated the degree to which a set of completed problems was a good attempt at the given operations. The patterns of represented answers either violated or did not violate principles of arithmetic fluency. Dixon et al. (2001) found that completed problems violating principles of arithmetic fluency were not rated as good attempts at the operations. Furthermore, based on the pattern of data, operations learned first (i.e., addition) demonstrated stronger mathematical representations than operations learned later (i.e., subtraction). Thus, Dixon et al. (2001) concluded that addition had a stronger mental representation than subtraction.

Peterson and Aller (1971) demonstrated that different arithmetic operations require different lengths of time for solution even when similar numbers are presented across problems. They compared addition, subtraction, and addition with negative numbers. Simple addition and multiplication problems were solved more quickly than negative addition or subtraction problems. Peterson and Aller (1971) suggested that memory facts for various types of mathematical operations might vary in accessibility, requiring longer searches for less accessible facts such as those involved in negative addition and subtraction.

Dixon et al. (2001) suggested that an important factor for generating mathematical solutions is a person's conceptual representation of the problem. This conceptual representation is a qualitative representation of the relationship among the objects in the problem. Therefore, people might select a solution technique for a given mathematical operation based on the structural match of the operation to their representation of the operation and this may specifically apply to the addition of negative numbers.

Conceptually, operations that include negative numbers may be most difficult to solve. According to the phylogenetic hypothesis of the origin of our comprehension of numerosities, the cognitive representation of the mental number line cannot represent negative numbers as well as positive numbers. This is because it is not possible to experience negative representations (Fischer, 2003). As such, negative numbers might be difficult to manipulate as they represent 'fictive' numbers (Fischer, 2003).

In the present research we examined adults' latencies and error rates on both simple subtraction and negative addition problems. In contrast to previous research examining negative numbers in polynomials (Peterson & Aller, 1971; Vlassis, 2004), we addressed the issue of negative numbers in simple addition. We hypothesized that adults are not familiar with seeing simple subtraction in a negative addition format and therefore expected that people would solve problems of this form slower than simple subtraction problems. Furthermore, we predicted that people would solve simple subtraction problems faster than simple negative addition problems as the former operation is learned before the latter. This expectancy was in accord with Dixon et al. (2001), who demonstrated the effects of learning orders on operations.

Adults were presented with problems in the form of simple subtraction (a – b) and simple addition of negative numbers a + (-b) on a paper-and-pencil test and were asked to generate answers. Response times were recorded and answers were verified for accuracy.

Method

Participants

Twenty-four adults voluntarily participated in this experiment, 13 men and 11 women. The participants were friends and family members of the principal investigators. All participants were over eighteen years of age. Volunteers were treated in accordance with the “Ethical Principles of Psychologists and Code of Conduct” (American Psychological Association, 1992).

Materials

Simple addition of negative numbers. The problem set was composed of 20 combinations of single-digit addition problems in the form a + (-b). The small addition condition consisted of 10 combinations of addends 2-4 and augends 1-3 (e.g., 3 + (-2)). The large negative addition condition consisted of 10 combinations of addends 6-9 and augends 5-8 (e.g., 8 + (-5)). Small and large simple negative addition conditions were presented on separate lists.

Simple subtraction problems. The problem set was composed of 20 combinations of single-digit subtraction problems in the form a – b. The small subtraction condition consisted of 10 small problems with minuends 2-4 and subtrahends 1-3 (e.g., 4 – 2). The large subtraction condition consisted of 10 combinations of minuends 6-9 and subtrahends 5-8 (e.g., 9 – 5). Small and large simple subtraction problems were presented on separate lists.

The order in which participants completed four lists (i.e., small negative addition, large negative addition, small subtraction, and large subtraction) was counterbalanced so that each participant was examined on a different order of the conditions. Furthermore, all conditions consisted of the same problems. Therefore, all 24 participants were administered the same problem lists in different orders.

Procedure

Each participant was individually tested in a single session lasting not more than 15 minutes. Before the session began, each participant was told that the purpose of the study was to determine response times and error rates for adults when solving simple arithmetic problems with and without brackets. Problems were presented on sheets of paper in four conditions: small negative addition, large negative addition, small subtraction, and large subtraction. Problems for each condition appeared on a separate sheet and were denoted as a trial. The order of the conditions was counterbalanced across participants. Participants were told the following instructions:

> You will be presented with four sheets of paper. Each piece of paper contains a set of 10 mathematical questions involving addition and subtraction. All sheets of paper will be presented face down. When instructed, please turn over your sheet of paper. Then proceed to answer the questions in the order they appear. Please let your examiner know when you are done with the piece of paper so that you may be presented with the next set of questions.

The experimenter initiated each trial by saying, “go.” Following the experimenter’s signal, each participant flipped over their trial sheet and recorded their answers. Upon completion of each trial the participant told the experimenter they were “done” and response time for that trial was recorded. Response times for each condition were recorded using a stopwatch and accuracy of responses was measured. Upon completion of each trial section the experimenter provided the participant with the next set of problems.

Results

There were very few errors made in this experiment. Of the 960 simple arithmetic problems, 3 responses were incorrect. Overall a 0.003% error rate occurred. As such, analysis and discussion of data will be mainly devoted to response times.

Problem Format

As hypothesized, participants were faster to solve problems in subtraction format (11 s) than negative addition format (13 s), $F\ (1, 23) = 14.20$, $MSE = 6.17$, $p < 0.005$.

Problem Size

Contrary to our expectations, there was no significant difference between time to solve large problems (12 s) and small problems (12 s), $F\ (1, 23) = 0.06$, $MSE = 5.60$, $p > 0.05$. A problem size effect might not have occurred due to our stimulus selection. All addends, augends, minuends, and subtrahends for problems of the form a + (-b) and a – b, respectively, did not exceed the integer nine. Consequently our range of numbers may have limited our expected problem size effect.

Interaction of Format and Problem Size

As shown in Figure 1, for response times, the interaction between format and problem size was significant, $F\ (1, 23) = 4.56$, $MSE = 3.23$, $p < 0.05$. Subtraction questions demonstrated a slight problem size effect. However for negative addition questions, the interaction between format and problem size demonstrated an unexpected reverse problem size effect.

Discussion

As hypothesized, participants solved problems of the form a – b faster than problems of the form a + (-b). Also, there was no significant difference between time to solve large problems and small problems in either condition (i.e., simple subtraction and negative addition). As such, there was no evidence of a significant problem size effect. However, for response times, there was a significant interaction between problem size and format. The subtraction questions demonstrated a slight problem size effect whereas negative addition questions demonstrated an unexpected reverse problem size effect.

Participants may have solved problems in the negative addition format slower due to their conceptual representation of the problems. Dixon et al. (2001) claim that an important factor for generating mathematical solutions is a person's conceptual representation of the problem. People might perform more poorly when dealing with negative numbers than positive numbers because negative numbers are associated with abstract representations (Fischer, 2003). According to the phylogenetic hypothesis of the origin of our comprehension of numerosities (Fischer, 2003), a cognitive representation of the mental number line cannot represent negative numbers as well as positive numbers. The lack of cognitive representation for negative numbers is due to the idea that it is impossible for one to experience negative representations.

Another reason for greater latencies in the negative addition format may be the brackets involved in these problems. One difference between a problem of the form a – b and a + (-b) are the brackets around the addend, -b. According to Ayres (2001), the cognitive load of brackets affects working memory. Ayres (2001) examined the effects of solving problems associated with expansion of two successive brackets (e.g., -3 (-4 – 5x) – 2(-3x – 4) = 12 + 15x + 6x + 8). Subsequently, he found that expansion of brackets affected the performance of participants on algebraic problems by affecting working memory. Although this type of problem is much more difficult than the problems in the current experiment, the notion of a greater cognitive load due to brackets is plausible.

There were a few participants (fewer than five) who solved the negative addition problems faster than the subtraction problems. These participants belonged to science disciplines and one explanation of these differential effects is that science students are more familiar with varying forms of mathematical operations than arts students. Therefore different forms of subtraction might affect students belonging to differing disciplines.

The most important variable in the area of mental arithmetic is that of the problem size effect. The problem size effect is the phenomenon whereby problems with larger numbers are solved more slowly than problems with smaller numbers (LeFevre et al., 1996). Contrary to our expectations, overall there was no significant difference found between time to solve large problems and small problems. A significant problem size effect may have not occurred due to our operational definition for small and large problems, whereby no integer was greater than 9. Specifically, both our small and large numbers consisted of one digit numbers. In contrast for subtraction problems, Seyler et al. (2003) differentiated between ‘small’ and ‘large’ problems by separating the problems into those with one- versus two-digit minuends (0-9 versus 10-18).

Furthermore in Seyler et al. (2003), all subtraction problems had one-digit subtrahends and remainders. Therefore examples of large problems, according to Seyler et al. (2003), are 18 – 9 or 10 - 9. Although both the former problems ‘look’ large, the latter problem has a remainder of 1and begs the validity of this question as being a ‘large’ problem. Upon analysis of our stimulus, we found that some of our large simple subtraction problems also consisted of differences being no greater than 1 (e.g., 9 – 8 =1). Therefore, it might be of interest to investigate whether or not the *difference* of a subtraction problem needs to be greater than a certain integer to be considered a 'small' or 'large' problem.

Contrary to our expectations, there was a slight *reverse* problem size effect for problems in the negative addition format. Therefore small negative addition problems were performed more slowly than large negative addition problems. Moreover, there was a greater difference in response time between small negative addition problems and small subtraction problems than large negative addition problems and large subtraction problems.

Campbell and Gunter (2002) found a similar reverse problem size effect upon examination of the tie (repeated operand) effect. The tie effect is the phenomenon whereby problems composed of a repeated operand (e.g., 6 + 6, 7 x 7), are typically solved faster and more accurately in comparison to non-time problems (e.g., 6 + 5, 7 x 8). Using simple multiplication problems, Campbell and Gunter presented tie problems in four formats: digit-digit (e.g., 4 x 4), digit-word (e.g., 4 x four), word-digit (e.g., four x 4) and word-word (e.g., four x four). For ties presented in mixed formats, the researchers found that the presentation eliminated the tie effect and also demonstrated an interference effect for mixed-format problems that increased response times.

We propose that a similar type of interference might have occurred in our experiment. Because simple addition problems are rarely seen in a negative addition format, some type of interference might have occurred whereby participants took longer to solve these problems. Furthermore, interference might have occurred because simple addition questions are generally solved by retrieval (LeFevre et al., 1996), therefore, causing greater interference than expected for a simple arithmetic problem. However, in the case of the large negative addition questions, participants may have been using other procedures for generating solutions to problems, therefore not allowing interference to make a significant difference in response time because there was not only ‘one’ method for answering the problem. When a problem is associated with a weak problem-procedure association and relatively strong problem-procedure associations, the likelihood of non-retrieval procedures is greater (LeFevre et al., 1996).

Conclusion

Arithmetic knowledge is of great importance to society and therefore examination of different mathematical formats may be beneficial for understanding cognitive processing. Furthermore, the significant difference for response times found between negative addition problems and subtraction problems implies that adults may not have clear conceptual knowledge of negative numbers. Researchers should examine why adults have more difficulty with negative numbers than with positive numbers so that the potential problem can be addressed in early education. Moreover, the absence of a problem size effect in this study requires greater examination to determine whether or not such an effect does exist. Modification of the stimulus in future research may prove useful for further investigation of the relationship between subtraction and negative addition.

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