

Distinct Distances in Graph Drawings

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Abstract

The *distance-number* of a graph G is the minimum number of distinct edge-lengths over all straight-line drawings of G in the plane. This definition generalises many well-known concepts in combinatorial geometry. We consider the distance-number of trees, graphs with no K_4^- -minor, complete bipartite graphs, complete graphs, and cartesian products. Our main results concern the distance-number of graphs with bounded degree. We prove that n -vertex graphs with bounded maximum degree and bounded treewidth have distance-number in $\mathcal{O}(\log n)$. To conclude such a logarithmic upper bound, both the degree and the treewidth need to be bounded. In particular, we construct graphs with treewidth 2 and polynomial distance-number. Similarly, we prove that there exist graphs with maximum degree 5 and arbitrarily large distance-number. Moreover, as Δ increases the existential lower bound on the distance-number of Δ -regular graphs tends to $\Omega(n^{0.864138})$.

1 Introduction

This paper initiates the study of the minimum number of distinct edge-lengths in a drawing of a given graph¹. A *degenerate drawing* of a graph G is a function that maps the

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¹We consider graphs that are simple, finite, and undirected. The vertex set of a graph G is denoted by $V(G)$, and its edge set by $E(G)$. A graph with n vertices, m edges and maximum degree at most Δ is an n -vertex, m -edge, degree- Δ graph. A graph in which every vertex has degree Δ is Δ -regular. For $S \subseteq V(G)$, let $G[S]$ be the subgraph of G induced by S , and let $G - S := G[V(G) \setminus S]$. For each vertex $v \in V(G)$, let $G - v := G - \{v\}$. Standard notation is used for graphs: complete graphs K_n , complete bipartite graphs $K_{m,n}$, paths P_n , and cycles C_n . A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. Throughout the paper, c is a positive constant. Of course, different occurrences of c might denote different constants.

vertices of G to distinct points in the plane, and maps each edge vw of G to the open straight-line segment joining the two points representing v and w . A *drawing* of G is a degenerate drawing of G in which the image of every edge of G is disjoint from the image of every vertex of G . That is, no vertex intersects the interior of an edge. In what follows, we often make no distinction between a vertex or edge in a graph and its image in a drawing.

The *distance-number* of a graph G , denoted by $\text{dn}(G)$, is the minimum number of distinct edge-lengths in a drawing of G . The *degenerate distance-number* of G , denoted by $\text{ddn}(G)$, is the minimum number of distinct edge-lengths in a degenerate drawing of G . Clearly, $\text{ddn}(G) \leq \text{dn}(G)$ for every graph G . Furthermore, if H is a subgraph of G then $\text{ddn}(H) \leq \text{ddn}(G)$ and $\text{dn}(H) \leq \text{dn}(G)$.

1.1 Background and Motivation

The degenerate distance-number and distance-number of a graph generalise various concepts in combinatorial geometry, which motivates their study.

A famous problem raised by Erdős [15] asks for the minimum number of distinct distances determined by n points in the plane². This problem is equivalent to determining the degenerate distance-number of the complete graph K_n . We have the following bounds on $\text{ddn}(K_n)$, where the lower bound is due to Katz and Tardos [25] (building on recent advances by Solymosi and Tóth [47], Solymosi et al. [46], and Tardos [50]), and the upper bound is due to Erdős [15].

Lemma 1 ([15, 25]). *The degenerate distance-number of K_n satisfies*

$$\Omega(n^{0.864137}) \leq \text{ddn}(K_n) \leq \frac{cn}{\sqrt{\log n}}.$$

Observe that no three points are collinear in a (non-degenerate) drawing of K_n . Thus $\text{dn}(K_n)$ equals the minimum number of distinct distances determined by n points in the plane with no three points collinear. This problem was considered by Szemerédi (see Theorem 13.7 in [37]), who proved that every such point set contains a point from which there are at least $\lceil \frac{n-1}{3} \rceil$ distinct distances to the other points. Thus we have the next result, where the upper bound follows from the drawing of K_n whose vertices are the points of a regular n -gon, as illustrated in Figure 1(a).

Lemma 2 (Szemerédi). *The distance-number of K_n satisfies*

$$\left\lceil \frac{n-1}{3} \right\rceil \leq \text{dn}(K_n) \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

Note that Lemmas 1 and 2 show that for every sufficiently large complete graph, the degenerate distance-number is strictly less than the distance-number. Indeed, $\text{ddn}(K_n) \in o(\text{dn}(K_n))$.

²For a detailed exposition on distinct distances in point sets refer to Chapters 10–13 of the monograph by Pach and Agarwal [37].

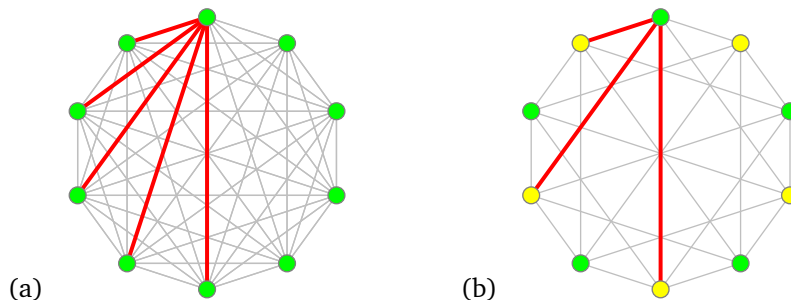


Figure 1: (a) A drawing of K_{10} with five edge-lengths, and (b) a drawing of $K_{5,5}$ with three edge-lengths.

Degenerate distance-number generalises another concept in combinatorial geometry. The *unit-distance graph* of a set S of points in the plane has vertex set S , where two vertices are adjacent if and only if they are at unit-distance; see [23, 35, 36, 39, 42, 45] for example. The famous Hadwiger-Nelson problem asks for the maximum chromatic number of a unit-distance graph. Every unit-distance graph G has $\text{ddn}(G) = 1$. But the converse is not true, since a degenerate drawing allows non-adjacent vertices to be at unit-distance. Figure 2 gives an example of a graph G with $\text{dn}(G) = \text{ddn}(G) = 1$ that is not a unit-distance graph. In general, $\text{ddn}(G) = 1$ if and only if G is isomorphic to a subgraph of a unit-distance graph.

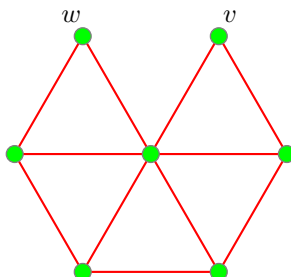


Figure 2: A graph with distance-number 1 that is not a unit-distance graph. In every mapping of the vertices to distinct points in the plane with unit-length edges, v and w are at unit-distance.

The maximum number of edges in a unit-distance graph is an old open problem. The best construction, due to Erdős [15], gives an n -vertex unit-distance graph with $n^{1+c/\log \log n}$ edges. The best upper bound on the number of edges is $cn^{4/3}$, due to Spencer et al. [48]. (Székely [49] found a simple proof for this upper bound based on the crossing lemma.)

More generally, many recent results in the combinatorial geometry literature provide upper bounds on the number of times the d most frequent inter-point distances can occur

between a set of n points. Such results are equivalent to upper bounds on the number of edges in an n -vertex graph with degenerate distance number d . This suggests the following extremal function. Let $\text{ex}(n, d)$ be the maximum number of edges in an n -vertex graph G with $\text{ddn}(G) \leq d$.

Since every graph G is the union of $\text{ddn}(G)$ subgraphs of unit-distance graphs, the above result by Spencer et al. [48] implies:

Lemma 3 (Spencer et al. [48]).

$$\text{ex}(n, d) \leq cdn^{4/3}.$$

Equivalently, the distance-numbers of every n -vertex m -edge graph G satisfy

$$\text{dn}(G) \geq \text{ddn}(G) \geq cmn^{-4/3}.$$

Results by Katz and Tardos [25] (building on recent advances by Solymosi and Tóth [47], Solymosi et al. [46], and Tardos [50]) imply:

Lemma 4 (Katz and Tardos [25]).

$$\text{ex}(n, d) \in \mathcal{O}(n^{1.457341} d^{0.627977}).$$

Equivalently, the distance-numbers of every n -vertex m -edge graph G satisfy

$$\text{dn}(G) \geq \text{ddn}(G) \in \Omega(m^{1.592412} n^{-2.320687}).$$

Note that Lemma 4 improves upon Lemma 3 whenever $\text{ddn}(G) > n^{1/3}$. Also note that Lemma 4 implies the lower bound in Lemma 2.

1.2 Our Results

The above results give properties of various graphs defined with respect to the inter-point distances of a set of points in the plane. This paper, which is more about graph drawing than combinatorial geometry, reverses this approach, and asks for a drawing of a given graph with few inter-point distances.

Our first results provide some general families of graphs, namely trees and graphs with no K_4^- -minor, that are unit-distance graphs (Section 2). Here K_4^- is the graph obtained from K_4 by deleting one edge. Then we give bounds on the distance-numbers of complete bipartite graphs (Section 3).

Our main results concern graphs of bounded degree (Section 4). We prove that for all $\Delta \geq 5$ there are degree- Δ graphs with unbounded distance-number. Moreover, for $\Delta \geq 7$ we prove a polynomial lower bound on the distance-number (of some degree- Δ graph) that tends to $\Omega(n^{0.864138})$ for large Δ . On the other hand, we prove that graphs with bounded degree and bounded treewidth have distance-number in $\mathcal{O}(\log n)$. Note that bounded treewidth alone does not imply a logarithmic bound on distance-number since $K_{2,n}$ has treewidth 2 and degenerate distance-number $\Theta(\sqrt{n})$ (see Section 3).

Then we establish an upper bound on the distance-number in terms of the bandwidth (Section 5). Then we consider the distance-number of the cartesian product of graphs (Section 6). We conclude in Section 7 with a discussion of open problems related to distance-number.

1.3 Higher-Dimensional Relatives

Graph invariants related to distances in higher dimensions have also been studied. Erdős, Harary, and Tutte [16] defined the *dimension* of a graph G , denoted by $\dim(G)$, to be the minimum integer d such that G has a degenerate drawing in \mathfrak{R}^d with straight-line edges of unit-length. They proved that $\dim(K_n) = n - 1$, the dimension of the n -cube is 2 for $n \geq 2$, the dimension of the Peterson graph is 2, and $\dim(G) \leq 2 \cdot \chi(G)$ for every graph G . (Here $\chi(G)$ is the *chromatic number* of G .) The dimension of complete 3-partite graphs and wheels were determined by Buckley and Harary [10].

The *unit-distance graph* of a set $S \subseteq \mathfrak{R}^d$ has vertex set S , where two vertices are adjacent if and only if they are at unit-distance. Thus $\dim(G) \leq d$ if and only if G is isomorphic to a subgraph of a unit-distance graph in \mathfrak{R}^d . Maehara [32] proved for all d there is a finite bipartite graph (which thus has dimension at most 4) that is not a unit-distance graph in \mathfrak{R}^d . This highlights the distinction between dimension and unit-distance graphs. Maehara [32] also proved that every finite graph with maximum degree Δ is a unit-distance graph in $\mathfrak{R}^{\Delta(\Delta^2-1)/2}$, which was improved to $\mathfrak{R}^{2\Delta}$ by Maehara and Rödl [33]. These results are in contrast to our result that graphs of bounded degree have arbitrarily large distance-number.

A graph is *d-realizable* if, for every mapping of its vertices to (not-necessarily distinct) points in \mathfrak{R}^p with $p \geq d$, there exists such a mapping in \mathfrak{R}^d that preserves edge-lengths. For example, K_3 is 2-realizable but not 1-realizable. Belk and Connelly [6] and Belk [5] proved that a graph is 2-realizable if and only if it has treewidth at most 2. They also characterized the 3-realizable graphs as those with no K_5 -minor and no $K_{2,2,2}$ -minor.

2 Some Unit-Distance Graphs

This section shows that certain families of graphs are unit-distance graphs. The proofs are based on the fact that two distinct circles intersect in at most two points. We start with a general lemma. A graph G is obtained by *pasting* subgraphs G_1 and G_2 on a cut-vertex v of G if $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{v\}$.

Lemma 5. *Let G be the graph obtained by pasting subgraphs G_1 and G_2 on a vertex v . Then:*

- (a) *if $\text{ddn}(G_1) = \text{ddn}(G_2) = 1$ then $\text{ddn}(G) = 1$, and*
- (b) *if $\text{dn}(G_1) = \text{dn}(G_2) = 1$ then $\text{dn}(G) = 1$.*

Proof. We prove part (b). Part (a) is easier. Let D_i be a drawing of G_i with unit-length edges. Translate D_2 so that v appears in the same position in D_1 and D_2 . A rotation of D_2 about v is *bad* if its union with D_1 is not a drawing of G . That is, some vertex in D_2 coincides with the closure of some edge of D_1 , or vice versa. Since G is finite, there are only finitely many bad rotations. Since there are infinitely many rotations, there exists a rotation that is not bad. That is, there exists a drawing of G with unit-length edges. \square

We have a similar result for unit-distance graphs.

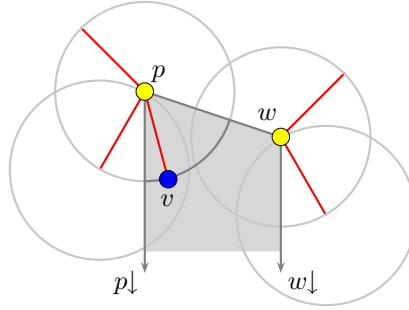


Figure 3: Illustration for the proof of Lemma 7

Lemma 6. *Let G_1 and G_2 be unit-distance graphs. Let G be the (abstract) graph obtained by pasting G_1 and G_2 on a vertex v . Then G is isomorphic to a unit-distance graph.*

Proof. The proof is similar to the proof of Lemma 5, except that we must ensure that the distance between vertices in $G_1 - v$ and vertices in $G_2 - v$ (which are not adjacent) is not 1. Again this will happen for only finitely many rotations. Thus there exists a rotation that works. \square

Since every tree can be obtained by pasting a smaller tree with K_2 , Lemma 6 implies that every tree is a unit-distance graph. The following is a stronger result.

Lemma 7. *Every tree T has a crossing-free³ drawing in the plane such that two vertices are adjacent if and only if they are unit-distance apart.*

Proof. For a point $v = (x(v), y(v))$ in the plane, let $v\downarrow$ be the ray from v to $(x(v), -\infty)$. We proceed by induction on n with the following hypothesis: Every tree T with n vertices has the desired drawing, such that the vertices have distinct x -coordinates, and for each vertex u , the ray $u\downarrow$ does not intersect T . The statement is trivially true for $n \leq 2$. For $n > 2$, let v be a leaf of T with parent p . By induction, $T - v$ has the desired drawing. Let w be a vertex of $T - v$, such that no vertex has its x -coordinate between $x(p)$ and $x(w)$. Thus the drawing of $T - v$ does not intersect the open region R of the plane bounded by the two rays $p\downarrow$ and $w\downarrow$, and the segment pw . Let A be the intersection of R with the unit-circle centred at p . Thus A is a circular arc. Place v on A , so that the distance from v to every vertex except p is not 1. This is possible since A is infinite, and there are only finitely many excluded positions on A (since A intersects a unit-circle centred at a vertex except p in at most two points). Since there are no elements of $T - v$ in R , there are no crossings in the resulting drawing and the induction invariants are maintained for all vertices of T . \square

Recall that K_4^- is the graph obtained from K_4 by deleting one edge.

Lemma 8. *Every 2-connected graph G with no K_4^- -minor is a cycle.*

³A drawing is *crossing-free* if no pair of edges intersect.

Proof. Suppose on the contrary that G has a vertex v of degree at least 3. Let x, y, z be the neighbours of v . There is an xy -path P avoiding v (since G is 2-connected) and avoiding z (since G is K_4^- -minor free). Similarly, there is an xz -path Q avoiding v . If x is the only vertex in both P and Q , then the cycle (x, P, y, v, z, Q) plus the edge xv is a subdivision of K_4^- . Now assume that P and Q intersect at some other vertex. Let t be the first vertex on P starting at x that is also in Q . Then the cycle (x, Q, z, v) plus the sub-path of P between x and t is a subdivision of K_4^- . This contradiction proves that G has no vertex of degree at least 3. Since G is 2-connected, G is a cycle, as desired. \square

Theorem 1. *Every K_4^- -minor-free graph G has a drawing such that vertices are adjacent if and only if they are unit-distance apart. In particular, G is isomorphic to a unit-distance graph and $\text{ddn}(G) = \text{dn}(G) = 1$.*

Proof. By Lemma 6, we can assume that G is 2-connected. Thus G is a cycle by Lemma 8. The result follows since C_n is a unit-distance graph (draw a regular n -gon). \square

3 Complete Bipartite Graphs

This section considers the distance-numbers of complete bipartite graphs $K_{m,n}$. Since $K_{1,n}$ is a tree, $\text{ddn}(K_{1,n}) = \text{dn}(K_{1,n}) = 1$ by Lemma 7. The next case, $K_{2,n}$, is also easily handled.

Lemma 9. *The distance-numbers of $K_{2,n}$ satisfy*

$$\text{ddn}(K_{2,n}) = \text{dn}(K_{2,n}) = \left\lceil \sqrt{\frac{n}{2}} \right\rceil.$$

Proof. Let $G = K_{2,n}$ with colour classes $A = \{v, w\}$ and B , where $|B| = n$. We first prove the lower bound, $\text{ddn}(K_{2,n}) \geq \left\lceil \sqrt{\frac{n}{2}} \right\rceil$. Consider a degenerate drawing of G with $\text{ddn}(G)$ edge-lengths. The vertices in B lie on the intersection of $\text{ddn}(G)$ concentric circles centered at v and $\text{ddn}(G)$ concentric circles centered at w . Since two distinct circles intersect in at most two points, $n \leq 2 \text{ddn}(G)^2$. Thus $\text{ddn}(K_{2,n}) \geq \left\lceil \sqrt{\frac{n}{2}} \right\rceil$.

For the upper bound, position v at $(-1, 0)$ and w at $(1, 0)$. As illustrated in Figure 4, draw $\left\lceil \sqrt{\frac{n}{2}} \right\rceil$ circles centered at each of v and w with radii ranging strictly between 1 and 2, such that the intersections of the circles together with v and w define a set of points with no three points collinear. (This can be achieved by choosing the radii iteratively, since for each circle C , there are finitely many forbidden values for the radius of C .) Each pair of non-concentric circles intersect in two points. Thus the number of intersection points is at least n . Placing the vertices of B at these intersection points results in a drawing of $K_{2,n}$ with $\left\lceil \sqrt{\frac{n}{2}} \right\rceil$ edge-lengths. \square

Now we determine $\text{ddn}(K_{3,n})$ to within a constant factor.

Lemma 10. *The degenerate distance-number of $K_{3,n}$ satisfies*

$$\left\lceil \sqrt{\frac{n}{2}} \right\rceil \leq \text{ddn}(K_{3,n}) \leq 3 \left\lceil \sqrt{\frac{n}{2}} \right\rceil - 1.$$

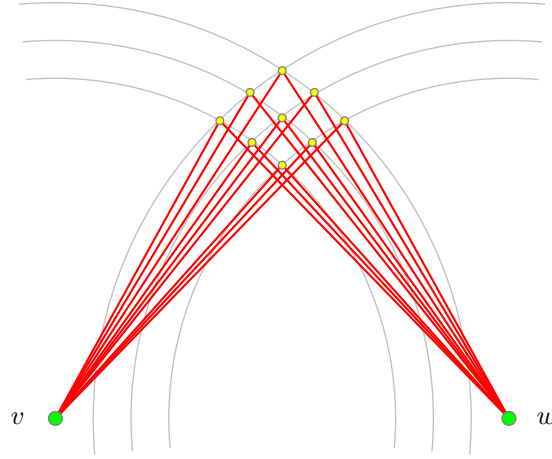


Figure 4: Illustration for the proof of Lemma 9.

Proof. The lower bound follows from Lemma 9 since $K_{2,n}$ is a subgraph of $K_{3,n}$.

Now we prove the upper bound. Let A and B be the colour classes of $K_{3,n}$, where $|A| = 3$ and $|B| = n$. Place the vertices in A at $(-1, 0)$, $(0, 0)$, and $(1, 0)$. Let $d := \lceil \sqrt{\frac{n}{2}} \rceil$. For $i \in [d]$, let

$$r_i := \sqrt{1 + \frac{i}{d+1}}.$$

Note that $1 < r_i < 2$. Let R_i be the circle centred at $(-1, 0)$ with radius r_i . For $j \in [d]$, let S_j be the circle centred at $(1, 0)$ with radius r_j . Observe that each pair of circles R_i and S_j intersect in exactly two points. Place the vertices in B at the intersection points of these circles. This is possible since $2d^2 \geq n$.

Let (x, y) and $(x, -y)$ be the two points where R_i and S_j intersect. Thus $(x+1)^2 + y^2 = r_i^2$ and $(x-1)^2 + y^2 = r_j^2$. It follows that

$$x^2 + y^2 = \frac{i}{d+1} + 2x = \frac{j}{d+1} - 2x.$$

Thus $2(x^2 + y^2) = \frac{i+j}{d+1}$. That is, the distance from (x, y) to $(0, 0)$ equals

$$\sqrt{\frac{i+j}{2d+2}},$$

which is the same distance from $(x, -y)$ to $(0, 0)$. Thus the distance from each vertex in B to $(0, 0)$ is one of $2d-1$ values (determined by $i+j$). The distance from each vertex in B to $(-1, 0)$ and to $(1, 0)$ is one of d values. Hence the degenerate distance-number of $K_{3,n}$ is at most $3d-1 = 3 \lceil \sqrt{\frac{n}{2}} \rceil - 1$. \square

Now consider the distance-number of a general complete bipartite graph.

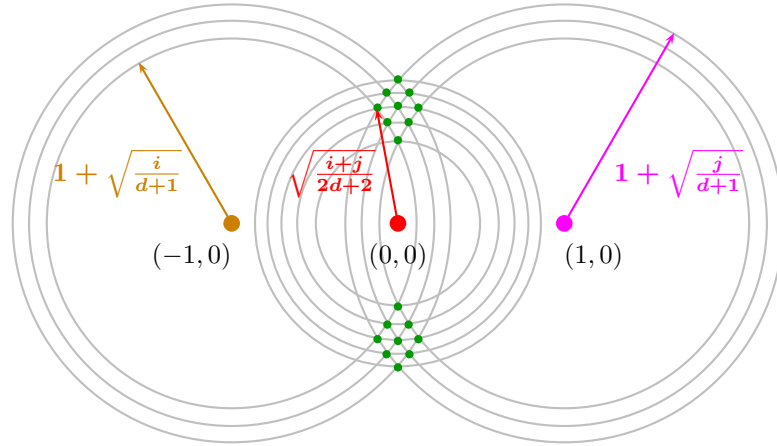


Figure 5: Illustration for the proof of Lemma 10.

Lemma 11. For all $n \geq m$, the distance-numbers of $K_{m,n}$ satisfy

$$\Omega\left(\frac{mn}{(m+n)^{1.457341}}\right)^{(1/0.627977)} \leq \text{ddn}(K_{m,n}) \leq \text{dn}(K_{m,n}) \leq \left\lceil \frac{n}{2} \right\rceil.$$

In particular,

$$\Omega(n^{0.864137}) \leq \text{ddn}(K_{n,n}) \leq \text{dn}(K_{n,n}) \leq \left\lceil \frac{n}{2} \right\rceil.$$

Proof. The lower bounds follow from Lemma 4. For the upper bound on $\text{dn}(K_{n,n})$, position the vertices on a regular $2n$ -gon $(v_1, v_2, \dots, v_{2n})$ alternating between the colour classes, as illustrated in Figure 1(b). In the resulting drawing of $K_{n,n}$, the number of edge-lengths is $|\{(i+j) \bmod n : v_i v_j \in E(K_{n,n})\}|$. Since $v_i v_j$ is an edge if and only if $i+j$ is odd, the number of edge-lengths is $\lceil \frac{n}{2} \rceil$. The upper bound on $\text{dn}(K_{n,m})$ follows since $K_{n,m}$ is a subgraph of $K_{n,n}$. \square

4 Bounded degree graphs

Lemma 9 implies that if a graph has two vertices with many common neighbours then its distance-number is necessarily large. Thus it is natural to ask whether graphs of bounded degree have bounded distance-number. This section provides a negative answer to this question.

4.1 Bounded degree graphs with $\Delta \geq 7$

This section proves that for all $\Delta \geq 7$ there are Δ -regular graphs with unbounded distance-number. Moreover, the lower bound on the distance-number is polynomial in the number of vertices. The basic idea of the proof is to show that there are more Δ -regular graphs

than graphs with bounded distance-number; see [4, 13, 14, 38] for other examples of this paradigm.

It will be convenient to count labelled graphs. Let $\mathcal{G}\langle n, \Delta \rangle$ denote the family of labelled Δ -regular n -vertex graphs. Let $\mathcal{G}\langle n, m, d \rangle$ denote the family of labelled n -vertex m -edge graphs with degenerate distance-number at most d . Our results follow by comparing a lower bound on $|\mathcal{G}\langle n, \Delta \rangle|$ with an upper bound on $|\mathcal{G}\langle n, m, d \rangle|$ with $m = \frac{\Delta n}{2}$, which is the number of edges in a Δ -regular n -vertex graph.

The lower bound in question is known. In particular, the first asymptotic bounds on the number of labelled Δ -regular n -vertex graphs were independently determined by Bender and Canfield [7] and Wormald [52]. McKay [34] further refined these results. We will use the following simple lower bound derived by Barát et al. [4] from the result of McKay [34].

Lemma 12 ([4, 7, 34, 52]). *For all integers $\Delta \geq 1$ and $n \geq c\Delta$, the number of labelled Δ -regular n -vertex graphs satisfies*

$$|\mathcal{G}\langle n, \Delta \rangle| \geq \left(\frac{n}{3\Delta}\right)^{\Delta n/2}.$$

The proof of our upper bound on $|\mathcal{G}\langle n, m, d \rangle|$ uses the following special case of the Milnor-Thom theorem by Rónyai et al. [43]. Let $\mathcal{P} = (P_1, P_2, \dots, P_t)$ be a sequence of polynomials on p variables over \mathfrak{R} . The *zero-pattern* of \mathcal{P} at $u \in \mathfrak{R}^p$ is the set $\{i : 1 \leq i \leq t, P_i(u) = 0\}$.

Lemma 13 ([43]). *Let $\mathcal{P} = (P_1, P_2, \dots, P_t)$ be a sequence of polynomials of degree at most $\delta \geq 1$ on $p \leq t$ variables over \mathfrak{R} . Then the number of zero-patterns of \mathcal{P} is at most $\binom{\delta t}{p}$.*

Recall that $\text{ex}(n, d)$ is the maximum number of edges in an n -vertex graph G with $\text{ddn}(G) \leq d$. Bounds on this function are given in Lemmas 3 and 4. Our upper bound on $|\mathcal{G}\langle n, m, d \rangle|$ is expressed in terms of $\text{ex}(n, d)$.

Lemma 14. *The number of labelled n -vertex m -edge graphs with $\text{ddn}(G) \leq d$ satisfies*

$$|\mathcal{G}\langle n, m, d \rangle| \leq \left(\frac{\mathbf{e}nd}{2}\right)^{2n+d} \binom{\text{ex}(n, d)}{m},$$

where \mathbf{e} is the base of the natural logarithm.

Proof. Let $V(G) = \{1, 2, \dots, n\}$ for every $G \in \mathcal{G}\langle n, m, d \rangle$. For every $G \in \mathcal{G}\langle n, m, d \rangle$, there is a point set

$$S(G) = \{(x_i(G), y_i(G)) : 1 \leq i \leq n\}$$

and a set of edge-lengths

$$L(G) = \{\ell_k(G) : 1 \leq k \leq d\},$$

such that G has a degenerate drawing in which each vertex i is represented by the point $(x_i(G), y_i(G))$ and the length of each edge in $E(G)$ is in $L(G)$. Fix one such degenerate drawing of G .

For all i, j, k with $1 \leq i < j \leq n$ and $1 \leq k \leq d$, and for every graph $G \in \mathcal{G}\langle n, m, d \rangle$, define

$$P_{i,j,k}(G) := (x_j(G) - x_i(G))^2 + (y_j(G) - y_i(G))^2 - \ell_k(G)^2.$$

Consider $\mathcal{P} := \{P_{i,j,k} : 1 \leq i < j \leq n, 1 \leq k \leq d\}$ to be a set of $\binom{n}{2}d$ degree-2 polynomials on the set of $2n + d$ variables $\{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n, \ell_1, \ell_2, \dots, \ell_d\}$. Observe that

$$P_{i,j,k}(G) = 0 \text{ if and only if the distance between vertices } i \text{ and } j \text{ in } \quad (\star)$$

the

degenerate drawing of G is $\ell_k(G)$.

Recall the well-known fact that $\binom{a}{b} \leq \left(\frac{ea}{b}\right)^b$. By Lemma 13 with $t = \binom{n}{2}d$, $\delta = 2$ and $p = 2n + d$, the number of zero-patterns determined by \mathcal{P} is at most

$$\binom{2\binom{n}{2}d}{2n+d} \leq \left(\frac{2e\binom{n}{2}d}{2n+d}\right)^{2n+d} < \left(\frac{en^2d}{2n+d}\right)^{2n+d} < \left(\frac{en^2d}{2n}\right)^{2n+d} = \left(\frac{end}{2}\right)^{2n+d}.$$

Fix a zero-pattern σ of \mathcal{P} . Let \mathcal{G}_σ be the set of graphs G in $\mathcal{G}\langle n, m, d \rangle$ such that σ is the zero-pattern of \mathcal{P} evaluated at G . To bound $|\mathcal{G}\langle n, m, d \rangle|$ we now bound $|\mathcal{G}_\sigma|$. Let H_σ be the graph with vertex set $V(H_\sigma) = \{1, \dots, n\}$ and edge set $E(H_\sigma)$ where $ij \in E(H_\sigma)$ if and only if $ij \in E(G)$ for some $G \in \mathcal{G}_\sigma$. Consider a degenerate drawing of an arbitrary graph $G \in \mathcal{G}_\sigma$ on the point set $S(G)$. By (\star) , $S(G)$ and $L(G)$ define a degenerate drawing of H with d edge-lengths. Thus $\text{ddn}(H_\sigma) \leq d$ and by assumption, $|E(H_\sigma)| \leq \text{ex}(n, d)$. Since every graph in \mathcal{G}_σ is a subgraph of H_σ , $|\mathcal{G}_\sigma| \leq \binom{|E(H_\sigma)|}{m}$. Therefore,

$$|\mathcal{G}\langle n, m, d \rangle| \leq \left(\frac{end}{2}\right)^{2n+d} \binom{|E(H_\sigma)|}{m} \leq \left(\frac{end}{2}\right)^{2n+d} \binom{\text{ex}(n, d)}{m},$$

as required. \square

By comparing the lower bound in Lemma 12 and the upper bound in Lemma 14 we obtain the following result.

Lemma 15. *Suppose that for some real numbers α and β with $\beta > 0$ and $1 < \alpha < 2 < \alpha + \beta$,*

$$\text{ex}(n, d) \in \mathcal{O}(n^\alpha d^\beta).$$

Then for every integer $\Delta > \frac{4}{2-\alpha}$, for all $\varepsilon > 0$, and for all sufficiently large $n > n(\alpha, \beta, \Delta, \varepsilon)$, there exists a Δ -regular n -vertex graph G with degenerate distance-number

$$\text{ddn}(G) > n^{\frac{2-\alpha}{\beta} - \frac{(2-\alpha+\beta)(4+2\varepsilon)}{\beta^2\Delta+4\beta}}.$$

Proof. In this proof, α, β, Δ and ε are fixed numbers satisfying the assumptions of the lemma. Let d be the maximum degenerate distance number of a graph in $\mathcal{G}\langle n, \Delta \rangle$. The result will follow by showing that for all sufficiently large $n > n(\alpha, \beta, \Delta, \varepsilon)$,

$$d > n^{\frac{2-\alpha}{\beta} - \frac{(2-\alpha+\beta)(4+2\varepsilon)}{\beta^2\Delta+4\beta}}.$$

By the definition of d , and since every Δ -regular n -vertex graph has $\frac{\Delta n}{2}$ edges, every graph in $\mathcal{G}\langle n, \Delta \rangle$ is also in $\mathcal{G}\langle n, \frac{\Delta n}{2}, d \rangle$. By Lemma 12 with $n \geq c\Delta$, and by Lemma 14,

$$\left(\frac{n}{3\Delta}\right)^{\Delta n/2} \leq |\mathcal{G}\langle n, \Delta \rangle| \leq |\mathcal{G}\langle n, \frac{\Delta n}{2}, d \rangle| \leq \left(\frac{end}{2}\right)^{2n+d} \binom{\text{ex}(n, d)}{\Delta n/2}.$$

Since $\text{ex}(n, d) \in \mathcal{O}(n^\alpha d^\beta)$, and since d is a function of n , there is a constant c such that $\text{ex}(n, d) \leq cn^\alpha d^\beta$ for sufficiently large n . Thus (and since $\binom{a}{b} \leq (\frac{ea}{b})^b$),

$$\left(\frac{n}{3\Delta}\right)^{\Delta n/2} \leq \left(\frac{end}{2}\right)^{2n+d} \binom{cn^\alpha d^\beta}{\Delta n/2} \leq \left(\frac{end}{2}\right)^{2n+d} \left(\frac{2ecn^\alpha d^\beta}{\Delta n}\right)^{\Delta n/2}.$$

Hence

$$n^{\Delta n} \leq 3^{\Delta n} \left(\frac{end}{2}\right)^{4n+2d} (2ecn^{\alpha-1}d^\beta)^{\Delta n}.$$

By Lemma 2, $d \leq \text{ddn}(K_n) \leq \frac{cn}{\sqrt{\log n}}$, implying $2d \leq \varepsilon n$ for all large $n > n(\varepsilon)$. Thus

$$n^\Delta \leq 3^\Delta \left(\frac{end}{2}\right)^{4+\varepsilon} (2ecn^{\alpha-1}d^\beta)^\Delta.$$

Hence

$$n^{(2-\alpha)\Delta-4-\varepsilon} \leq 3^\Delta \left(\frac{e}{2}\right)^{4+\varepsilon} (2ec)^\Delta d^{\beta\Delta+4+\varepsilon}.$$

Observe that $3^\Delta \left(\frac{e}{2}\right)^{4+\varepsilon} (2ec)^\Delta \leq n^\varepsilon$ for all large $n > n(\Delta, \varepsilon)$. Thus

$$n^{(2-\alpha)\Delta-4-2\varepsilon} \leq d^{\beta\Delta+4+\varepsilon}.$$

Hence

$$d \geq n^{\frac{(2-\alpha)\Delta-4-2\varepsilon}{\beta\Delta+4+\varepsilon}} = n^{\frac{2-\alpha}{\beta} - \frac{(2-\alpha+\beta)(4+\varepsilon)+\beta\varepsilon}{\beta(\beta\Delta+4+\varepsilon)}} > n^{\frac{2-\alpha}{\beta} - \frac{(2-\alpha+\beta)(4+2\varepsilon)}{\beta^2\Delta+4\beta}},$$

as required. \square

We can now state the main results of this section. By Lemma 3, the conditions of Lemma 15 are satisfied with $\alpha = \frac{4}{3}$ and $\beta = 1$; thus together they imply:

Theorem 2. *For every integer $\Delta \geq 7$, for all $\varepsilon > 0$, and for all sufficiently large $n > n(\Delta, \varepsilon)$, there exists a Δ -regular n -vertex graph G with degenerate distance-number*

$$\text{ddn}(G) > n^{\frac{2}{3} - \frac{20+10\varepsilon}{3\Delta+12}}.$$

By Lemma 4, the conditions of Lemma 15 are satisfied with $\alpha = 1.457341$ and $\beta = 0.627977$; thus together they imply:

Theorem 3. *For every integer $\Delta \geq 8$, for all $\varepsilon > 0$, and for all sufficiently large $n > n(\Delta, \varepsilon)$, there exists a Δ -regular n -vertex graph G with degenerate distance-number*

$$\text{ddn}(G) > n^{0.864138 - \frac{4.682544+2.341272\varepsilon}{0.394355\Delta+2.511908}}.$$

Note that the bound given in Theorem 3 is better than the bound in Theorem 2 for $\Delta \geq 17$.

4.2 Bounded degree graphs with $\Delta \geq 5$

Theorem 2 shows that for $\Delta \geq 7$ and for sufficiently large n , there is an n -vertex degree- Δ graph whose degenerate distance-number is at least polynomial in n . We now prove that the degenerate distance-number of degree-5 graphs can also be arbitrarily large. However, the lower bound we obtain in this case is polylogarithmic in n . The proof is inspired by an analogous proof about the slope-number of degree-5 graphs, due to Pach and Pálvölgyi [38].

Theorem 4. *For all $d \in \mathbb{N}$, there is a degree-5 graph G with degenerate distance-number $\text{ddn}(G) > d$.*

Proof. Consider the following degree-5 graph G . For $n \equiv 0 \pmod{6}$, let F be the graph with vertex set $\{v_1, v_2, \dots, v_n\}$ and edge set $\{v_i v_j : |i - j| \leq 2\}$. Let $S := \{v_i : i \equiv 1 \pmod{3}\}$. No pair of vertices in S are adjacent in F , and $|S| = \frac{n}{3}$ is even.

Let \mathcal{M} denote the set of all perfect matchings on S . For each perfect matching $M_k \in \mathcal{M}$, let $G_k := F \cup M_k$. As illustrated in Figure 6, let G be the disjoint union of all the G_k . Thus the number of connected components of G is $|\mathcal{M}|$, which is at least $(\frac{n}{9})^{n/6}$ by Lemma 12 with $\Delta = 1$. Here we consider perfect matchings to be 1-regular graphs. (It is remarkable that even with $\Delta = 1$, Lemma 12 gives such an accurate bound, since the actual number of matchings in S is $\sqrt{2}(\frac{n}{3e})^{n/6}$ ignoring lower order additive terms⁴.)

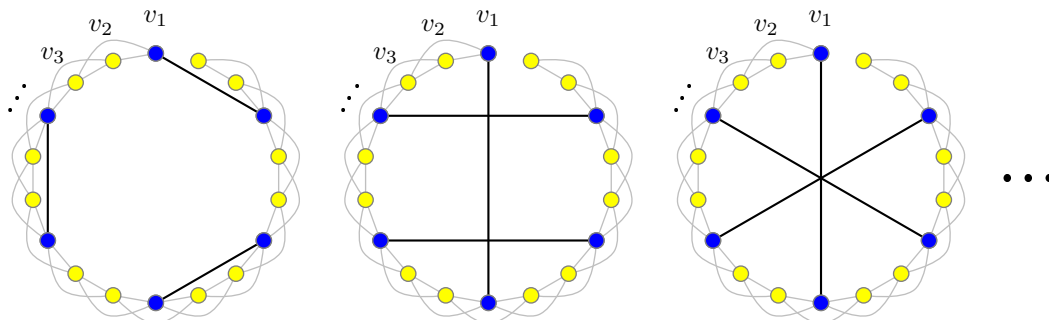


Figure 6: The graph G with $n = 18$.

Suppose, for the sake of contradiction, that for some constant d , for all $n \in \mathbb{N}$ such that $n \equiv 0 \pmod{6}$, G has a degenerate drawing D with at most d edge-lengths.

Label the edges of G that are in the copies of F by their length in D . Let $\ell_k(i, j)$ be the label of the edge $v_i v_j$ in the copy of F in the component G_k of G . This defines a

⁴For even n , let $f(n)$ be the number of perfect matchings of $[n]$. Here we determine the asymptotics of f . In every such matching, n is matched with some number in $[n - 1]$, and the remaining matching is isomorphic to a perfect matching of $[n - 2]$. Every matching obtained in this way is distinct. Thus $f(n) = (n - 1) \cdot f(n - 2)$, where $f(2) = 1$. Hence $f(n) = (n - 1)!! = (n - 1)(n - 3)(n - 5) \dots 1$, where $!!$ is the double factorial function. Now $(2n - 1)!! = \frac{(2n)!}{2^n n!}$. Thus $f(n) = \frac{n!}{2^{n/2} (n/2)!} \approx \sqrt{2} (\frac{n}{e})^{n/2}$ by Stirling's Approximation.

labelling of the components of G . Since F has $2n - 3$ edges and each edge in F receives one of d labels, there are at most d^{2n-3} distinct labellings of the components of G .

Let D_k be the degenerate drawing of G_k obtained from D by a translation and rotation so that v_1 is at $(0, 0)$ and v_2 is at $(\ell_k(1, 2), 0)$. We say that two components G_q and G_r of G *determine the same set of points* if for all $i \in [n]$, the vertex v_i in D_q is at the same position as the vertex v_i in D_r .

Partition the components of G into the minimum number of parts such that all the components in each part have the same labelling and determine the same set of points.

Observe that two components of G with the same labelling do not necessarily determine the same set of points. However, the number of point sets determined by the components with a given labelling can be bounded as follows. For each component G_k of G , v_1 is at $(0, 0)$ and v_2 is at $(\ell_k(i, j), 0)$ in D_k . Thus for a fixed labelling, the positions of v_1 and v_2 in D_k are determined. Now for $i \geq 3$, in each component G_k , the vertex v_i is positioned in D_k at the intersection of the circle of radius $\ell_k(i - 1, i)$ centered at v_{i-1} and the circle of radius $\ell_k(i - 2, i)$ centered at v_{i-2} . Thus there are at most two possible locations for v_i (for a fixed labelling). Hence the components with the same labelling determine at most 2^{n-2} distinct points sets. Therefore the number of parts in the partition is at most $d^{2n-3} \cdot 2^{n-2} < (2d^2)^n$.

Finally, we bound the number of components in each part, R , of the partition. Let H_R be the graph with vertex set $V(H_R) = \{v_1, \dots, v_n\}$ where $v_i v_j \in E(H_R)$ if and only if $v_i v_j \in E(G_k)$ for some component $G_k \in R$. Since the graphs in R determine the same set of points, the union of the degenerate drawings D_k , over all $G_k \in R$, determines a degenerate drawing of H_R with d edge-lengths. Thus $\text{ddn}(H_R) \leq d$ and by Lemma 3, $|E(H_R)| \leq c d n^{4/3}$ for some constant $c > 0$. Every component in R is a subgraph of H_R , and any two components in R differ only by the choice of a matching on S . Each such matching has $\frac{n}{6}$ edges. Thus the number of components of G in R is at most

$$\binom{|E(H_R)|}{n/6} \leq \binom{c d n^{4/3}}{n/6} \leq \left(\frac{e c d n^{4/3}}{n/6} \right)^{n/6} \leq (6 e c d)^{n/6} n^{n/18}.$$

Hence $|\mathcal{M}| < (2d^2)^n \cdot (6ecd)^{n/6} n^{n/18}$, and by the lower bound on $|\mathcal{M}|$ from the start of the proof,

$$\left(\frac{n}{9} \right)^{n/6} \leq |\mathcal{M}| < (2d^2)^n \cdot (6ecd)^{n/6} n^{n/18}.$$

The desired contradiction follows for all $n \geq (3456ecd^{13})^{3/2}$. □

4.3 Graphs with bounded degree and bounded treewidth

This section proves a logarithmic upper bound on the distance-number of graphs with bounded degree and bounded treewidth. Treewidth is an important parameter in Robertson and Seymour's theory of graph minors and in algorithmic complexity (see the surveys [8, 41]). It can be defined as follows. A graph G is a k -tree if either $G = K_{k+1}$, or G has a vertex v whose neighbourhood is a clique of order k and $G - v$ is a k -tree. For

example, every 1-tree is a tree and every tree is a 1-tree. Then the *treewidth* of a graph G is the minimum integer k for which G is a subgraph of a k -tree. The *pathwidth* of G is the minimum k for which G is a subgraph of an interval⁵ graph with no clique of order $k + 2$. Note that an interval graph with no $(k + 2)$ -clique is a special case of a k -tree, and thus the treewidth of a graph is at most its pathwidth.

Lemma 7 shows that (1-)trees have bounded distance-number. However, this is not true for 2-trees since $K_{2,n}$ has treewidth (and pathwidth) at most 2. By Theorem 3, there are n -vertex graphs of bounded degree with distance-number approaching $\Omega(n^{0.864138})$. On the other hand, no polynomial lower bound holds for graphs of bounded degree and bounded treewidth, as shown in the following theorem.

Theorem 5. *Let G be a graph with n vertices, maximum degree Δ , and treewidth k . Then the distance-number of G satisfies*

$$\text{dn}(G) \in \mathcal{O}(\Delta^4 k^3 \log n).$$

To prove Theorem 5 we use the following lemma, the proof of which is readily obtained by inspecting the proof of Lemma 8 in [14]. An H -partition of a graph G is a partition of $V(G)$ into vertex sets V_1, \dots, V_t such that H is the graph with vertex set $V(H) := \{1, \dots, t\}$ where $ij \in E(H)$ if and only if there exists $v \in V_i$ and $w \in V_j$ such that $v_i v_j \in E(G)$. The *width* of an H -partition is $\max\{|V_i| : 1 \leq i \leq t\}$.

Lemma 16 ([14]). *Let H be a graph admitting a drawing D with s distinct edge-slopes and ℓ distinct edge-lengths. Let G be a graph admitting an H -partition of width w . Then the distance-number of G satisfies*

$$\text{dn}(G) \leq slw(w - 1) + \left\lfloor \frac{w}{2} \right\rfloor + \ell.$$

Sketch of Proof. The general approach is to start with D and then replace each vertex of H by a sufficiently scaled down and appropriate rotated copy of the drawing of K_w on a regular w -gon. The only difficulty is choosing the rotation and the amount by which to scale the w -gons so that we obtain a (non-degenerate) drawing of G . Refer to [14] for the full proof. \square

Proof of Theorem 5. Let w be the minimum width of a T -partition of G in which T is a tree. The best known upper bound is $w \leq \frac{5}{2}(k + 1)(\frac{7}{2}\Delta(G) - 1)$, which was obtained by Wood [51] using a minor improvement to a similar result by an anonymous referee of the paper by Ding and Oporowski [12]. For each vertex $x \in V(T)$, there are at most $w\Delta$ edges of G incident to vertices mapped to x . Hence we can assume that T is a forest with maximum degree $w\Delta$, as otherwise there is an edge of T with no edge of G mapped to it, in which case the edge of T can be deleted. Similarly, T has at most n vertices. Scheffler [44] proved that T has pathwidth at most $\log(2n + 1)$; see [8]. Dujmović et al.

⁵A graph G is an *interval graph* if each vertex $v \in V(G)$ can be assigned an interval $I_v \subset \mathfrak{R}$ such that $I_w \cap I_v \neq \emptyset$ if and only if $vw \in E(V)$.

[14] proved that every tree T with pathwidth $p \geq 1$ has a drawing with $\max\{\Delta(T) - 1, 1\}$ slopes and $2p - 1$ edge-lengths. Thus T has a drawing with at most $\Delta w - 1$ slopes and at most $2 \log(2n + 1) - 1$ edge-lengths. By Lemma 16,

$$\text{dn}(G) \leq (\Delta w - 1)(2 \log(2n + 1) - 1)w(w - 1) + \left\lfloor \frac{w}{2} \right\rfloor + 2 \log(2n + 1) - 1,$$

which is in $\mathcal{O}(\Delta w^3 \log n) \subseteq \mathcal{O}(\Delta^4 k^3 \log n)$. \square

Corollary 1. *Any n -vertex graph with bounded degree and bounded treewidth has distance-number $\mathcal{O}(\log n)$.*

Since a path has a drawing with one slope and one edge-length, Lemma 16 with $s = \ell = 1$ implies that every graph G with a P -partition of width k for some path P has distance-number $\text{dn}(G) \leq k(k - \frac{1}{2}) + 1$.

5 Bandwidth

This section finds an upper bound on the distance-number in terms of the bandwidth. Let G be a graph. A *vertex ordering* of G is a bijection $\sigma : V(G) \rightarrow \{1, 2, \dots, |V(G)|\}$. The *width* of σ is defined to be $\max\{|\sigma(v) - \sigma(w)| : vw \in E(G)\}$. The *bandwidth* of G , denoted by $\text{bw}(G)$, is the minimum width of a vertex ordering of G . The *cyclic width* of σ is defined to be $\max\{\min\{|\sigma(v) - \sigma(w)|, n - |\sigma(v) - \sigma(w)|\} : vw \in E(G)\}$. The *cyclic bandwidth* of G , denoted by $\text{cbw}(G)$, is the minimum cyclic width of a vertex ordering of G ; see [11, 20, 28, 30, 53]. Clearly $\text{cbw}(G) \leq \text{bw}(G)$ for every graph G .

Lemma 17. *For every graph G ,*

$$\text{dn}(G) \leq \text{cbw}(G) \leq \text{bw}(G).$$

Proof. Given a vertex ordering σ of an n -vertex G , position the vertices of G on a regular n -gon in the order σ . We obtain a drawing of G in which the length of each edge vw is determined by

$$\min\{|\sigma(v) - \sigma(w)|, n - |\sigma(v) - \sigma(w)|\}.$$

Thus the number of edge-lengths equals

$$|\{\min\{|\sigma(v) - \sigma(w)|, n - |\sigma(v) - \sigma(w)|\} : vw \in E(G)\}|,$$

which is at most the cyclic width of σ . The result follows. \square

Corollary 2. *The distance-number of every n -vertex degree- Δ planar graph G satisfies*

$$\text{dn}(G) \leq \frac{15n}{\log_{\Delta} n}.$$

Proof. Böttcher et al. [9] proved that $\text{bw}(G) \leq \frac{15n}{\log_{\Delta} n}$. The result follows from Lemma 17. \square

6 Cartesian Products

This section discusses the distance-number of cartesian products of graphs. For graphs G and H , the *cartesian product* $G \square H$ is the graph with vertex set $V(G \square H) := V(G) \times V(H)$, where (v, w) is adjacent to (p, q) if and only if (1) $v = p$ and wq is an edge of H , or (2) $w = q$ and vp is an edge of G .

Thus $G \square H$ is the grid-like graph with a copy of G in each row and a copy of H in each column. Type (1) edges form copies of H , and type (2) edges form copies of G . For example, $P_n \square P_n$ is the planar grid, and $C_n \square C_n$ is the toroidal grid.

The cartesian product is associative and thus multi-dimensional products are well defined. For example, the d -dimensional product $K_2 \square K_2 \square \dots \square K_2$ is the d -dimensional hypercube Q_d . It is well known that Q_d is a unit-distance graph. Horvat and Pisanski [24] proved that the cartesian product operation preserves unit-distance graphs. That is, if G and H are unit-distance graphs, then so is $G \square H$, as illustrated in Figure 7. The following theorem generalises this result.

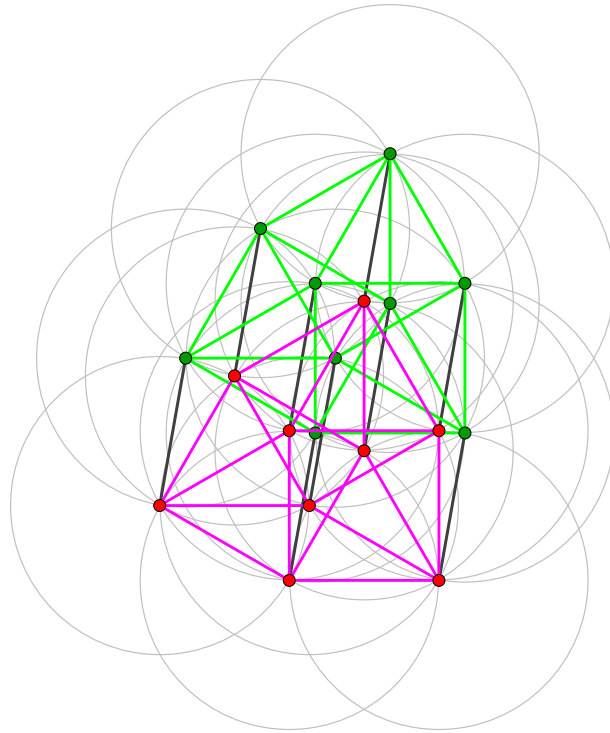


Figure 7: A unit-distance drawing of $K_3 \square K_3 \square K_2$

Theorem 6. *For all graphs G and H , the distance-numbers of $G \square H$ satisfy*

$$\max\{\text{ddn}(G), \text{ddn}(H)\} \leq \text{ddn}(G \square H) \leq \text{ddn}(G) + \text{ddn}(H) - 1, \text{ and}$$

$$\max\{\text{dn}(G), \text{dn}(H)\} \leq \text{dn}(G \square H) \leq \text{dn}(G) + \text{dn}(H) - 1.$$

Proof. The lower bounds follow since G and H are subgraphs of $G \square H$. We prove the upper bound for $\text{dn}(G \square H)$. The proof for $\text{ddn}(G \square H)$ is simpler.

Fix a drawing of G with $\text{dn}(G)$ edge-lengths. Let $(x(v), y(v))$ be the coordinates of each vertex v of G in this drawing. Fix a drawing of H with $\text{dn}(H)$ edge-lengths, scaled so that one edge-length in the drawing of G coincides with one edge-length in the drawing of H . Let α be a real number in $[0, 2\pi)$. Let $(x_\alpha(w), y_\alpha(w))$ be the coordinates of each vertex w of G in this drawing of H rotated by α degrees about the origin.

Position vertex (v, w) in $G \square H$ at $(x(v) + x_\alpha(w), y(v) + y_\alpha(w))$. This mapping preserves edge-lengths. In particular, the length of a type-(1) edge $(v, u)(v, w)$ equals the length of the edge uw in H , and the length of a type-(2) edge $(u, v)(w, v)$ equals the length of the edge uw in G . Thus for each α , the mapping of $G \square H$ has $\text{dn}(G) + \text{dn}(H) - 1$ edge-lengths.

It remains to prove that for some α the mapping of $G \square H$ is a drawing. That is, no vertex intersects the closure of an incident edge. An angle α is *bad* for a particular vertex/edge pair of $G \square H$ if that vertex intersects the closure of that edge in the mapping with rotation α .

Observe that the trajectory of a vertex (v, w) of $G \square H$ (taken over all α) is a circle centred at $(x(v), y(v))$ with radius $\text{dist}_H(0, w)$.

Now for distinct points p and q and a line ℓ , there are only two angles α such that the rotation of p around q by an angle of α contains ℓ (since the trajectory of p is a circle that only intersects ℓ in two places), and there are only two angles α such that the rotation of ℓ around q by an angle of α contains p .

It follows that there are finitely many bad values of α for a particular vertex/edge pair of $G \square H$. Hence there are finitely many bad values of α in total. Hence some value of α is not bad for every vertex/edge pair in $G \square H$. Hence D_α is a valid drawing of $G \square H$. \square

Note that Loh and Teh [31] proved a result analogous to Theorem 6 for dimension.

Let G^d be the d -fold cartesian product of a graph G . The same construction used in Theorem 6 proves the following:

Theorem 7. *For every graph G and integer $d \geq 1$, the distance-numbers of G^d satisfy*

$$\text{ddn}(G^d) = \text{ddn}(G) \quad \text{and} \quad \text{dn}(G^d) = \text{dn}(G).$$

7 Open Problems

We conclude by mentioning some of the many open problems related to distance-number.

- What is $\text{dn}(K_n)$? Pach and Agarwal [37] write that “it can be conjectured” that $\text{dn}(K_n) = \lfloor \frac{n}{2} \rfloor$. That is, every set of n points in general position determine at least $\lfloor \frac{n}{2} \rfloor$ distinct distances. Note that Altman [1, 2] proved this conjecture for points in convex position.
- What is the relationship between distance-number and degenerate distance-number? In particular, is there a function f such that $\text{dn}(G) \leq f(\text{ddn}(G))$ for every graph G ?

- Theorems 2, 3 and 4 establish a lower bound for the distance-number of bounded degree graphs. But no non-trivial upper bound is known. Do n -vertex graphs with bounded degree have distance-number in $o(n)$?
- Outerplanar graphs have distance-number in $\mathcal{O}(\Delta^4 \log n)$ by Theorem 5. Do outerplanar graphs (with bounded degree) have bounded (degenerate) distance-number?
- Non-trivial lower and upper bounds on the distance-numbers are not known for many other interesting graph families including: degree-3 graphs, degree-4 graphs, 2-degenerate graphs with bounded degree, graphs with bounded degree and bounded pathwidth.
- As described in Section 1.1, determining the maximum number of times the unit-distance can appear among n points in the plane is a famous open problem. We are unaware if the following apparently simpler tasks have been attempted: Determine the maximum number of times the unit-distance can occur among n points in the plane such that no three are collinear. Similarly, determine the maximum number of edges in an n -vertex graph G with $\text{dn}(G) = 1$.
- Determining the maximum chromatic number of unit-distance graphs in \mathbb{R}^d is a well-known open problem. The best known upper bound of $(3 + o(1))^d$ is due to Larman and Rogers [29]. Exponential lower bounds are known [17, 40]. Unit-distance graphs in the plane are 7-colourable [19], and thus $\chi(G) \leq 7^{\text{dn}(G)}$. Can this bound be improved?
- Degenerate distance-number is not bounded by any function of dimension since $K_{n,n}$ has dimension 4 and unbounded degenerate distance-number. On the other hand, $\text{dim}(G) \leq 2 \cdot \chi(G) \leq 2 \cdot 7^{\text{dn}(G)}$. Is $\text{dim}(G)$ bounded by a polynomial function of $\text{dn}(G)$?
- Every planar graph has a crossing-free drawing. A long standing open problem involving edge-lengths, due to Harborth et al. [21, 22, 26], asks whether every planar graph has a crossing-free drawing in which the length of every edge is an integer. Geelen et al. [18] recently answered this question in the affirmative for cubic planar graphs. Archdeacon [3] extended this question to nonplanar graphs and asked what is the minimum d such that a given graph has a crossing-free drawing in \mathbb{R}^d with integer edge-lengths. Note that every n -vertex graph has such a drawing in \mathbb{R}^{n-1} .
- The *slope number* of a graph G , denoted by $\text{sn}(G)$, is the minimum number of edge-slopes over all drawings of G . Dujmović et al. [13] established results concerning the slope-number of planar graphs. Keszegh et al. [27] proved that degree-3 graphs have slope-number at most 5. On the other hand, Barát et al. [4] and Pach and Pálvölgyi [38] independently proved that there are 5-regular graphs with arbitrarily large slope number. Moreover, for $\Delta \geq 7$, Dujmović et al. [14] proved that there are n -vertex degree- Δ graphs whose slope number is at least $n^{1 - \frac{1}{\Delta+4}}$. The proofs of these

results are similar to the proofs of Theorems 2, 3 and 4. Given that Theorem 5 also depends on slopes, it is tempting to wonder if there is a deeper connection between slope-number and distance-number. For example, is there a function f such that $\text{sn}(G) \leq f(\Delta(G), \text{dn}(G))$ and/or $\text{dn}(G) \leq f(\text{sn}(G))$ for every graph G . Note that some dependence on $\Delta(G)$ is necessary since $\text{sn}(K_{1,n}) \rightarrow \infty$ but $\text{dn}(K_{1,n}) = 1$.

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