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LIOUVILLE'S SEXTENARY QUADRATIC FORMS $x^2 + y^2 + z^2 + t^2 + 2u^2 + 2v^2$, $x^2 + y^2 + 2z^2 + 2t^2 + 2u^2 + 2v^2$ AND $x^2 + 2y^2 + 2z^2 + 2t^2 + 2u^2 + 4v^2$

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Abstract

Liouville's asserted formulae for the number of representations of a positive integer by each of the sextenary quadratic forms $x^2 + y^2 + z^2 + t^2 + 2u^2 + 2v^2$, $x^2 + y^2 + 2z^2 + 2t^2 + 2u^2 + 2v^2$ and $x^2 + 2y^2 + 2z^2 + 2t^2 + 2u^2 + 4v^2$ are proved.

1. Introduction

Let $\mathbb{Z}$ and $\mathbb{N}$ denote the sets of integers and positive integers, respectively. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{N}$ and $n \in \mathbb{N}_0$

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we set

$$N(a_1, a_2, a_3, a_4, a_5, a_6; n)$$

$$= \text{card}\{(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{Z}^6 \mid n = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 + a_5x_5^2 + a_6x_6^2\}. \tag{1.1}$$

Clearly

$$N(a_1, a_2, a_3, a_4, a_5, a_6; 0) = 1. \tag{1.2}$$

Formulae for $N(a_1, \ldots, a_6; n)$ for $(a_1, \ldots, a_6) = (1, 1, 1, 1, 2, 2)$, $(1, 1, 2, 2, 2, 2)$ and $(1, 2, 2, 2, 2, 4)$ valid for all $n \in \mathbb{N}$ were originally conjectured by Liouville ([6], [7], [8]). It is the purpose of this paper to give simple proofs of these formulae.

2. Determination of $N(1, 1, 1, 1, 2, 2; n)$, $N(1, 1, 2, 2, 2, 2; n)$ and $N(1, 2, 2, 2, 2, 4; n)$

Following Ramanujan [3, p. 6] we set

$$\varphi(q) = \sum_{n\in\mathbb{Z}} q^{n^2}, \quad \psi(q) = \sum_{n\in\mathbb{N}_0} q^{n(n+1)/2}. \tag{2.1}$$

The basic identities satisfied by φ and ψ are

$$\varphi(q) + \varphi(-q) = 2\varphi(q^4), \tag{2.2}$$

$$\varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2), \tag{2.3}$$

$$\varphi(q)\varphi(-q) = \varphi^2(-q^2), \tag{2.4}$$

$$\varphi(q) - \varphi(-q) = 4q\psi(q^8), \tag{2.5}$$

$$\varphi(q)\psi(q^2) = \psi^2(q), \tag{2.6}$$

see [3, pp. 15, 71, 72]. From (2.2)-(2.6) we obtain the following identities.

Theorem 2.1.

(i) $\varphi^6(q) = \varphi^2(q)\varphi^4(-q) + 16q\varphi^2(q)\psi^4(q^2)$.

(ii) $\varphi^4(q)\varphi^2(q^2) = \frac{1}{2}\varphi^2(q)\varphi^4(-q) + \frac{1}{2}\varphi^4(q)\varphi^2(-q) + 8q\varphi^2(q)\psi^4(q^2).$

(iii) $\varphi^2(q)\varphi^4(q^2) = \frac{1}{2}\varphi^2(q)\varphi^4(-q) + \frac{1}{2}\varphi^4(q)\varphi^2(-q) + 4q\varphi^2(q)\psi^4(q^2).$

(iv) $\varphi(q)\varphi^4(q^2)\varphi(q^4) = \varphi^2(q^4)\varphi^4(-q^4) + 2q\varphi^2(q)\psi^4(q^2).$

Proof. It is convenient to set

$$a := \varphi(q), \quad b := \varphi(-q).$$

Then, from (2.2)-(2.6), we obtain

$$\varphi^2(q^2) = \frac{1}{2}(a^2 + b^2), \quad \varphi(q^4) = \frac{1}{2}(a + b),$$

$$\varphi^2(-q^2) = ab, \quad \varphi^4(-q^4) = \frac{1}{2}(a^3b + ab^3),$$

$$q\psi^8(q) = \frac{1}{16}(a^8 - a^4b^4), \quad q\psi^4(q^2) = \frac{1}{16}(a^4 - b^4),$$

$$q\psi^2(q^4) = \frac{1}{8}(a^2 - b^2), \quad q\psi(q^8) = \frac{1}{4}(a - b).$$

(i) We have

$$\varphi^2(q)\varphi^4(-q) + 16q\varphi^2(q)\psi^4(q^2) = a^2b^4 + a^2(a^4 - b^4) = a^6 = \varphi^6(q).$$

(ii) We have

$$\frac{1}{2}\varphi^2(q)\varphi^4(-q) + \frac{1}{2}\varphi^4(q)\varphi^2(-q) + 8q\varphi^2(q)\psi^4(q^2)$$

$$= \frac{1}{2}a^2b^4 + \frac{1}{2}a^4b^2 + \frac{1}{2}a^2(a^4 - b^4)$$

$$= \frac{1}{2}a^4(a^2 + b^2)$$

$$= \varphi^4(q)\varphi^2(q^2).$$

(iii) We have

$$\frac{1}{2}\varphi^2(q)\varphi^4(-q) + \frac{1}{2}\varphi^4(q)\varphi^2(-q) + 4q\varphi^2(q)\psi^4(q^2)$$

$$= \frac{1}{2}a^2b^4 + \frac{1}{2}a^4b^2 + \frac{1}{4}a^2(a^4 - b^4)$$

$$= \frac{1}{4}a^2(a^2 + b^2)^2$$

$$= \varphi^2(q)\varphi^4(q^2).$$

(iv) We have

$$\varphi^2(q^4)\varphi^4(-q^4) + 2q\varphi^2(q)\psi^4(q^2) = \frac{1}{8}(a+b)^2(a^3b + ab^3) + \frac{1}{8}a^2(a^4 - b^4)$$

$$= \frac{1}{8}a(a+b)(a^2 + b^2)^2$$

$$= \varphi(q)\varphi^4(q^2)\varphi(q^4).$$

This completes the proof of Theorem 2.1. □

In [2, Theorems 2.4, 2.5] we used a formula of Carlitz [4, eq. (1.3), p. 168] to prove the following formulae:

$$\sum_{n=1}^{\infty} G_4(n)q^n = q\prod_{n=1}^{\infty}\frac{(1-q^{2n})^6(1-q^{4n})^4}{(1-q^n)^4} \tag{2.7}$$

and

$$1 - 4\sum_{n=1}^{\infty} H_4(n)q^n = \prod_{n=1}^{\infty}\frac{(1-q^n)^4(1-q^{2n})^6}{(1-q^{4n})^4}, \tag{2.8}$$

where for $n \in \mathbb{N}$ the arithmetic functions G_4 and H_4 are defined by

$$G_4(n) := \sum_{\substack{d\in\mathbb{N}\\ d\mid n}}\left(\frac{-4}{n/d}\right)d^2, \quad H_4(n) := \sum_{\substack{d\in\mathbb{N}\\ d\mid n}}\left(\frac{-4}{d}\right)d^2. \tag{2.9}$$

If $m \notin \mathbb{N}$, then we define $G_4(m) = H_4(m) = 0$. Jacobi [5] has shown in his famous paper on elliptic functions that

$$\varphi(q) = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^5}{(1-q^n)^2(1-q^{4n})^2}, \quad \psi(q) = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^2}{(1-q^n)}. \tag{2.10}$$

From (2.10) we deduce

$$\psi(q^2) = \prod_{n=1}^{\infty} \frac{(1-q^{4n})^2}{(1-q^{2n})}, \quad \varphi(-q) = \prod_{n=1}^{\infty} \frac{(1-q^n)^2}{(1-q^{2n})}. \tag{2.11}$$

Hence, from (2.7), (2.8), (2.10) and (2.11), we obtain the following result.

Theorem 2.2.

(i) $\sum_{n=1}^{\infty} G_4(n)q^n = q\varphi^2(q)\psi^4(q^2)$.

(ii) $1 - 4\sum_{n=1}^{\infty} H_4(n)q^n = \varphi^2(q)\varphi^4(-q)$.

We are now ready to determine $N(1, 1, 1, 1, 2, 2; n)$, $N(1, 1, 2, 2, 2, 2; n)$ and $N(1, 2, 2, 2, 2, 4; n)$ in terms of $G_4(n)$ and $H_4(n)$ for all $n \in \mathbb{N}$. In addition we reprove Jacobi's formula for $N(1, 1, 1, 1, 1, 1; n)$, see for example [1].

Theorem 2.3. *For* $n \in \mathbb{N}$

(i) $N(1, 1, 1, 1, 1, 1; n) = 16G_4(n) - 4H_4(n)$.

(ii) $N(1, 1, 1, 1, 2, 2; n) = 8G_4(n) - 2(1 + (-1)^n)H_4(n)$.

(iii) $N(1, 1, 2, 2, 2, 2; n) = 4G_4(n) - 2(1 + (-1)^n)H_4(n)$.

(iv) $N(1, 2, 2, 2, 2, 4; n) = 2G_4(n) - 4H_4(n/4)$.

Proof. (i) By Theorems 2.1(i) and 2.2 we have

$$\begin{aligned}\sum_{n=0}^{\infty} N(1, 1, 1, 1, 1, 1; n)q^n &= \varphi^6(q) \\ &= \varphi^2(q)\varphi^4(-q) + 16q\varphi^2(q)\psi^4(q^2)\end{aligned}$$

$$= 1 - 4\sum_{n=1}^{\infty} H_4(n)q^n + 16\sum_{n=1}^{\infty} G_4(n)q^n.$$

Equating coefficients of q^n $(n \in \mathbb{N})$, we obtain

$$N(1, 1, 1, 1, 1, 1; n) = 16G_4(n) - 4H_4(n).$$

(ii) By Theorems 2.1(ii) and 2.2 we have

$$\sum_{n=0}^{\infty} N(1, 1, 1, 1, 2, 2; n)q^n$$

$$= \varphi^4(q)\varphi^2(q^2)$$

$$= \frac{1}{2}\varphi^2(q)\varphi^4(-q) + \frac{1}{2}\varphi^4(q)\varphi^2(-q) + 8q\varphi^2(q)\psi^4(q^2)$$

$$= \frac{1}{2}\left(1 - 4\sum_{n=1}^{\infty} H_4(n)q^n\right) + \frac{1}{2}\left(1 - 4\sum_{n=1}^{\infty} H_4(n)(-q)^n\right) + 8\sum_{n=1}^{\infty} G_4(n)q^n.$$

Equating coefficients of q^n $(n \in \mathbb{N})$, we obtain

$$N(1, 1, 1, 1, 2, 2; n) = 8G_4(n) - 2(1 + (-1)^n)H_4(n).$$

(iii) By Theorems 2.1(iii) and 2.2 we deduce

$$\sum_{n=0}^{\infty} N(1, 1, 2, 2, 2, 2; n)q^n$$

$$= \varphi^2(q)\varphi^4(q^2)$$

$$= \frac{1}{2}\varphi^2(q)\varphi^4(-q) + \frac{1}{2}\varphi^4(q)\varphi^2(-q) + 4q\varphi^2(q)\psi^4(q^2)$$

$$= \frac{1}{2}\left(1 - 4\sum_{n=1}^{\infty} H_4(n)q^n\right) + \frac{1}{2}\left(1 - 4\sum_{n=1}^{\infty} H_4(n)(-q)^n\right) + 4\sum_{n=1}^{\infty} G_4(n)q^n.$$

Equating coefficients of q^n $(n \in \mathbb{N})$, we obtain

$$N(1, 1, 2, 2, 2, 2; n) = 4G_4(n) - 2(1 + (-1)^n)H_4(n).$$

(iv) By Theorems 2.1(iv) and 2.2 we have

$$\sum_{n=0}^{\infty} N(1, 2, 2, 2, 2, 4; n)q^n = \varphi(q)\varphi^4(q^2)\varphi(q^4)$$

$$= \varphi^2(q^4)\varphi^4(-q^4) + 2q\varphi^2(q)\psi^4(q^2)$$

$$= 1 - 4\sum_{n=1}^{\infty} H_4(n)q^{4n} + 2\sum_{n=1}^{\infty} G_4(n)q^n.$$

Equating coefficients of q^n $(n \in \mathbb{N})$, we obtain

$$N(1, 2, 2, 2, 2, 4; n) = 2G_4(n) - 4H_4(n/4).$$

This completes the proof of Theorem 2.3. □

We close by giving an alternative formulation of Theorem 2.3.

Theorem 2.4. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha N$, *where* $\alpha \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $N \equiv 1 \pmod 2$. *Let*

$$N = \prod_{p \mid N} p^{\alpha_p}$$

be the prime factorization of N. Then

(i) $$N(1, 1, 1, 1, 1, 1; n) = \left(2^{2\alpha+4} - 4\left(\frac{-4}{N}\right)\right)N^2 \prod_{p \mid N} \frac{1 - \left(\frac{-4}{p}\right)p^{-2\alpha_p - 2}}{1 - \left(\frac{-4}{p}\right)p^{-2}}.$$

(ii) $N(1, 1, 1, 1, 2, 2; n)$

$$= \left(2^{2\alpha+3} - 2(1 + (-1)^{2^\alpha})\left(\frac{-4}{N}\right)\right)N^2 \prod_{p \mid N} \frac{1 - \left(\frac{-4}{p}\right)^{\alpha_p+1} p^{-2\alpha_p - 2}}{1 - \left(\frac{-4}{p}\right)p^{-2}}.$$

(iii) $N(1, 1, 2, 2, 2, 2; n)$

$$= \left(2^{2\alpha+2} - 2(1 + (-1)^{2^\alpha})\left(\frac{-4}{N}\right)\right)N^2 \prod_{p \mid N} \frac{1 - \left(\frac{-4}{p}\right)^{\alpha_p+1} p^{-2\alpha_p - 2}}{1 - \left(\frac{-4}{p}\right)p^{-2}}.$$

(iv) $N(1, 2, 2, 2, 2, 4; n)$

$$= \left(2^{2\alpha+1} - 4c(\alpha)\left(\frac{-4}{N}\right)\right) N^2 \prod_{p \mid N} \frac{1 - \left(\frac{-4}{p}\right)^{\alpha_p+1} p^{-2\alpha_p-2}}{1 - \left(\frac{-4}{p}\right) p^{-2}},$$

where

$$c(\alpha) = \begin{cases} 0, & \text{if } \alpha = 0, 1, \\ 1, & \text{if } \alpha \geq 2. \end{cases}$$

Proof. It is easy to show that

$$G_4(n) = 2^{2\alpha} G_4(N), \quad H_4(n) = H_4(N), \quad G_4(N) = \left(\frac{-4}{N}\right) H_4(N).$$

Next, as $\left(\frac{-4}{d}\right) d^2$ is a multiplicative function of d, we have

$$G_4(N) = \sum_{d \mid N} \left(\frac{-4}{N/d}\right) d^2$$

$$= N^2 \sum_{d \mid N} \left(\frac{-4}{d}\right) \frac{1}{d^2}$$

$$= N^2 \prod_{p \mid N} \left(\sum_{\beta=0}^{\alpha_p} \left(\frac{-4}{p}\right)^{\beta} p^{-2\beta} \right)$$

$$= N^2 \prod_{p \mid N} \frac{1 - \left(\frac{-4}{p}\right)^{\alpha_p+1} p^{-2\alpha_p-2}}{1 - \left(\frac{-4}{p}\right) p^{-2}}.$$

Thus

$$G_4(n) = 2^{2\alpha} N^2 \prod_{p \mid N} \frac{1 - \left(\frac{-4}{p}\right)^{\alpha_p+1} p^{-2\alpha_p-2}}{1 - \left(\frac{-4}{p}\right) p^{-2}}$$

and

$$H_4(n) = \left(\frac{-4}{N}\right) N^2 \prod_{p \mid N} \frac{1 - \left(\frac{-4}{p}\right)^{\alpha_p + 1} p^{-2\alpha_p - 2}}{1 - \left(\frac{-4}{p}\right) p^{-2}}.$$

Using these two formulae in Theorem 2.3 we obtain Theorem 2.4. □

For $u \in \mathbb{N}$ and p an odd prime we have

$$1 - \left(\frac{-4}{p}\right) p^{-2u} \geq 1 - \frac{1}{p^{2u}} \geq 1 - \frac{1}{p^2} \geq \frac{8}{9} > 0$$

so

$$\prod_{p \mid N} \frac{1 - \left(\frac{-4}{p}\right) p^{-2\alpha_p - 2}}{1 - \left(\frac{-4}{p}\right) p^{-2}} > 0.$$

Also it is easy to check that each of

$$2^{2\alpha+4} - 4\left(\frac{-4}{N}\right),\ 2^{2\alpha+3} - 2(1 + (-1)^{2^\alpha})\left(\frac{-4}{N}\right),$$

$$2^{2\alpha+2} - 2(1 + (-1)^{2^\alpha})\left(\frac{-4}{N}\right),\ 2^{2\alpha+1} - 4c(\alpha)\left(\frac{-4}{N}\right)$$

is positive. Thus we have the following result.

Corollary 2.1. *For* $n \in \mathbb{N}$

$$N(1, 1, 1, 1, 1, 1; n) > 0,$$

$$N(1, 1, 1, 1, 2, 2; n) > 0,$$

$$N(1, 1, 2, 2, 2, 2; n) > 0,$$

$$N(1, 2, 2, 2, 2, 4; n) > 0.$$

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