

# A formula for the bivariate map asymptotics constants in terms of the univariate map asymptotics constants

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## Abstract

The parameters  $t_g$ ,  $p_g$ ,  $t_g(r)$  and  $p_g(r)$  appear in the asymptotics for a variety of maps on surfaces and embeddable graphs. In this paper we express  $t_g(r)$  in terms of  $t_g$  and  $p_g(r)$  in terms of  $p_g$ .

## 1 Introduction

The concepts in this paragraph will be made precise in the following paragraphs. The parameters  $t_g$  and  $p_g$  arise in the univariate asymptotic enumeration of a variety of maps on surfaces and the parameters  $t_g(r)$  and  $p_g(r)$  arise in the corresponding bivariate asymptotics for maps as well as embeddable graphs. The original recursions for these parameters make it extremely difficult to compute them for higher genus surfaces. In contrast, the other parameters in the asymptotics are usually easily determined. Recently a simple recursion has been obtained for  $t_g$  and another conjectured for  $p_g$ . In this paper, we obtain simple expressions for the bivariate parameters  $t_g(r)$  and  $p_g(r)$  in terms of the corresponding univariate parameters.

A *map* is a connected graph  $G$  embedded in a surface  $S$  (a closed 2-manifold) such that all components of  $S - G$  are simply connected regions, which are called *faces*. Loops and multiple edges are allowed in  $G$ . A map is *rooted* if an edge is distinguished together with a direction on the edge and a side of the edge. The exact enumeration of various types of maps on the sphere (or, equivalently, the plane) was carried out by Tutte and his students (see [28] for a survey) in the 1960s via his device of rooting. Beginning in the 1980s, Tutte's approach was used for the asymptotic enumeration of maps on general surfaces [3, 4, 9, 11, 16, 17, 18, 19]. A matrix integral approach was initiated by 't

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Hooft (see [25] for various connections with quantum gravity, representation theory, and algebraic geometry). Let  $T_g(n)$  ( $P_g(n)$ ) be the number of rooted  $n$ -edge maps on the orientable surface of genus  $g$  (non-orientable surface with  $2g$  cross-caps). In 1986 Bender and Canfield showed that, for each fixed  $g$  and as  $n \rightarrow \infty$ ,

$$T_g(n) \sim t_g n^{5(g-1)/2} 12^n, \quad P_g(n) \sim p_g n^{5(g-1)/2} 12^n, \quad (1)$$

where  $t_g$  and  $p_g$  are positive constants which can be computed by complicated recursions. In 1988 Bender and Wormald [11] derived similar asymptotic formulas for 2-connected maps in which the constants  $t_g$  and  $p_g$  also appear.

In 1993, the author [18] showed that many natural families of maps satisfy asymptotic formulas similar to (1) in which the same constants  $t_g$  and  $p_g$  appear in the coefficients. So in some sense  $t_g$  and  $p_g$  are universal constants. There is a nice connection between  $t_g$  and Painlevé I ODE, and this connection seems to be well-known in the quantum physics community. However, there are doubts as to whether the proofs of the relevant results in the physics literature are mathematically rigorous. See, e.g., [25, Section 3.6] and [14, p. 29] for some related information. It is also worth mentioning that conjecture (74) stated in [14, p. 29] follows immediately from [19, Thm. 1.4].

Recently, using representation theory and KP-hierarchy, Goulden and Jackson [22] derived a remarkably simple recursion for the numbers of rooted triangulations of orientable surfaces. Let  $C_{n,g}$  be the number of rooted  $2n$ -face triangulations (or, by duality,  $2n$ -vertex cubic maps) of an orientable surface of genus  $g$ . Define  $H_{n,g} = (3n+2)C_{n,g}$  for  $n \geq 1$ ,  $g \geq 0$ , and

$$H_{-1,0} = 1/2, \quad H_{0,0} = 2 \quad \text{and} \quad H_{-1,g} = H_{0,g} = 0 \quad \text{for} \quad g \neq 0.$$

Goulden and Jackson [22] showed that, for  $(n, g) \neq (-1, 0)$ ,

$$H_{n,g} = \frac{4(3n+2)}{n+1} \left( n(3n-2)H_{n-2,g-1} + \sum_{i=-1}^{n-1} \sum_{h=0}^g H_{i,h} H_{n-2-i,g-h} \right). \quad (2)$$

Bender et al. [7] used this recursion to derive a simple recursion for  $t_g$  which leads to an asymptotic formula for  $t_g$ . This asymptotic formula for  $t_g$  was used in [20] to settle a conjecture of 't Hooft about analyticity of free energy. Let

$$f_g = 24^{-3/2} 6^{g/2} \Gamma\left(\frac{5g-1}{2}\right) t_g.$$

It was shown in [7] that

$$f_g = \frac{\sqrt{6}}{96} (5g-4)(5g-6) f_{g-1} + 6\sqrt{6} \sum_{h=1}^{g-1} f_h f_{g-h}, \quad f_0 = -\frac{\sqrt{6}}{72},$$

and hence the generating function  $f(z) = \sum_{g \geq 1} f_g z^g$  satisfies the following second order nonlinear ODE: (note there are two typos in the ODE given in [7])

$$f(z) = 6\sqrt{6}(f(z))^2 + \frac{\sqrt{6}}{96} z \left( 25z^2 f''(z) + 25z f'(z) - f(z) + \frac{\sqrt{6}}{72} \right).$$

Garoufalidis et al. [20] noticed that the above ODE is Painlevé I in disguise. More precisely, they noticed that

$$a_g = -\frac{72}{\sqrt{6}} \left(\frac{2}{\sqrt{6}}\right)^g f_g = -2^{g-2} \Gamma\left(\frac{5g-1}{2}\right) t_g$$

satisfies the following recursion

$$a_g = \frac{(5g-4)(5g-6)}{48} a_{g-1} - \frac{1}{2} \sum_{h=1}^{g-1} a_h a_{g-h}, \quad a_0 = 1, \quad (3)$$

and the formal series  $w(z) = \sum_{g \geq 0} a_g z^{-(5g-1)/2}$  satisfies the following Painlevé I:

$$w''(z) = 6w^2(z) - 6z.$$

This recursion was studied by Joshi and Kitaev [24] in the context of Painlevé I, and they derived the following full asymptotic expansion:

$$a_g \sim \frac{S}{\pi} A^{-2g+1/2} \Gamma(2g-1/2) \left( 1 + \sum_{l \geq 1} \frac{\mu_l A^l}{\prod_{k=1}^l (2g-k-1/2)} \right),$$

where

$$A = \frac{8\sqrt{3}}{5}, \quad S = -\frac{1}{2\sqrt{\pi}} 3^{1/4},$$

and  $\mu_l$  can be computed recursively using

$$\mu_l = \frac{5}{16\sqrt{3}l} \left( \frac{192}{25} \sum_{k=0}^{l-1} \mu_k a_{(l-k+1)/2} - \left(l - \frac{9}{10}\right) \left(l - \frac{1}{10}\right) \mu_{l-1} \right), \quad \mu_0 = 1.$$

In the above (and below), it is understood that  $a_j = 0$  when  $j$  is not an integer.

Based on evidence from quantum physics, Garoufalidis and Mariño [21] conjectured that

$$p_g = \frac{1}{2^{g-2} \Gamma\left(\frac{5g-3}{2}\right)} v_{2g-1}, \quad (4)$$

where  $v_g$  satisfies

$$v_g = \frac{1}{2\sqrt{3}} \left( -3a_{g/2} + \frac{5g-6}{2} v_{g-1} + \sum_{k=1}^{g-1} v_k v_{g-k} \right),$$

and  $a_j$  is defined by (3). In [21], a nice asymptotic formula was also derived for  $v_g$  using the above recursion for  $v_g$  and the asymptotic expression for  $a_g$ .

In [5, 12], interesting connections were shown between  $t_g$  and the  $g$ th moment of some random variables defined on trees.

In 1993, Bender, Canfield, and Richmond [4] derived a bivariate version of formula (1). Let  $T_g(i, j)$  ( $P_g(i, j)$ ) be the number of rooted maps, with  $i$  faces and  $j$  vertices, on the orientable surface of genus  $g$  (non-orientable surface with  $2g$  cross-caps). They showed

$$T_g(i, j) \sim t_g(r)(ij)^{5g/4-3/2}u_0^{-i}v_0^{-j}, \quad P_g(i, j) \sim p_g(r)(ij)^{5g/4-3/2}u_0^{-i}v_0^{-j}, \quad (5)$$

where

$$u_0 = \frac{r^3(2+r)}{4(1+r+r^2)^2}, \quad v_0 = \frac{1+2r}{4(1+r+r^2)^2}, \quad (6)$$

and  $r > 0$  is determined by  $j/i$  using the equation

$$\frac{j}{i} = \frac{1+2r}{r^2(2+r)}.$$

For each  $r > 0$ ,  $t_g(r)$  and  $p_g(r)$  are positive constants which can be computed by complicated recursions (which are given in sections 2 and 3 below).

Our main result in this paper is the following.

**Theorem 1** *Define*

$$\begin{aligned} c(r) &= \frac{r^3(1+2r)(2+r)}{32\sqrt{\pi}}(4+7r+4r^2)^{-1/2}(1+r+r^2)^{-7/2}, \\ d(r) &= 32\sqrt{3}r^{-7/2}(1+r+r^2)^4(1+r)^{3/2}(2+r)^{-5/4}(1+2r)^{-5/4}. \end{aligned}$$

*Then*

$$t_g(r) = c(r)[d(r)]^g t_g, \quad (7)$$

$$p_g(r) = c(r)[d(r)]^g p_g. \quad (8)$$

We note that the above formulas easily lead to asymptotic formulas for  $t_g(r)$  and  $p_g(r)$  (as  $g \rightarrow \infty$ ), using the corresponding asymptotic formulas for  $t_g$  and  $p_g$ .

Finally we mention that  $t_g(r)$  and  $p_g(r)$  also appear in the asymptotic expressions for the numbers of 2-connected and 3-connected maps with  $i$  faces and  $j$  vertices [6]. Recently there have been considerable interest in enumerating graphs with a given genus (see, e.g., [8, 23, 26, 27]). Let  $G(S; n)$  be the number of labelled graphs (no loops or multiple edges) with  $n$  vertices which are embeddable in a surface  $S$ . In [26], McDiarmid established the exponential growth rate of  $G(S; n)/n!$  by showing that, for each fixed surface  $S$ ,

$$\lim_{n \rightarrow \infty} (G(S; n)/n!)^{1/n} = \gamma$$

for some positive constant  $\gamma$  which is independent of  $S$ . The algebraic growth rate of  $G(S; n)$  was only established very recently. Bender and Gao [6] and Chapuy et al. [13] independently showed that

$$G(S; n)/n! \sim c(S)n^{(5g-7)/2}\gamma^n, \quad (n \rightarrow \infty)$$

where  $g = 1 - \chi(S)/2$  with  $\chi(S)$  being the Euler characteristic of the surface  $S$ , and  $c(S)$  is a positive constant depending on  $S$ . In [6], it was shown that

$$c(S) = \begin{cases} AB^g t_g(r_0) & \text{: when } S \text{ is the orientable surface of genus } g, \\ AB^g p_g(r_0) & \text{: when } S \text{ is the non-orientable surface with } 2g \text{ cross-caps,} \end{cases}$$

where  $r_0, A$ , and  $B$  are positive constants which are independent of  $S$ . Furthermore,  $t_g(r)$  and  $p_g(r)$  also appear in the asymptotic expressions for the numbers of  $k$ -connected ( $0 \leq k \leq 3$ ) labelled graphs of genus  $g$  with respect to vertices and edges.

Our approach is similar to that used in [18]. Using an appropriate normalizing factor, we can show that the complicated recursions satisfied by  $t_g$  and  $t_g(r)$  (similarly for  $p_g$  and  $p_g(r)$ ) are equivalent. The main difference is that here we are comparing recursions for  $t_g(r)$  ( $p_g(r)$ ), which are bivariate in the sense that they involve a second parameter  $r$ , with the univariate recursions for  $t_g$  ( $p_g$ ), whereas in [18] all recursions are univariate. As a result, our normalizing factor used in this paper is slightly more sophisticated and involves the second parameter  $r$ .

## 2 Connection between $t_g(r)$ and $t_g$

In this section we prove Theorem 1 for orientable surfaces. Our approach will be similar to that used in [18]. We will show that the recursions satisfied by  $t_g(r)$  can be normalized to match those satisfied by  $t_g$ . We need to recall some definitions and notation from [3, 4].

Let  $\hat{M}_g(x, y, I)$  be the generating function for rooted maps on the orientable surface of genus  $g$ , where  $x$  marks the number of edges,  $y$  marks the root face degree, and each  $z_i, i \in I$ , marks the degree of the  $i$ th distinguished face. For

$$f = \frac{5 - \sqrt{1 - 12x}}{4 + 2x}, \quad \boldsymbol{\alpha} = (\alpha_i)_{i \in I}, \quad \text{and} \quad |\boldsymbol{\alpha}| = \sum_{i \in I} \alpha_i,$$

define

$$\hat{M}_g^{(n)}(x, I, \boldsymbol{\alpha}) = \frac{\partial^{n+|\boldsymbol{\alpha}|}}{\partial y^n \prod_{i \in I} \partial z_i^{\alpha_i}} \Big|_{y=z_i=f}.$$

We note that our  $\hat{M}_g^{(n)}(x, I, \boldsymbol{\alpha})$  is the same as  $\hat{H}_g^{(n)}(x, I, \boldsymbol{\alpha})$  used in [3].

In the following,

$$F(x) \approx c(1 - x/x_0)^a \quad (\text{as } x \rightarrow x_0)$$

means that  $F(x)$  is analytic in the region  $\{x : |x| < x_0 + \delta\} - [x_0, x_0 + \delta]$  for some small  $\delta > 0$ , and it can be written as

$$F(x) = p(x) + c(1 - x/x_0)^a + o((1 - x/x_0)^a), \quad (\text{as } x \rightarrow x_0)$$

where  $p(x)$  is a polynomial in  $x$ ,  $x_0, c \neq 0$ , and  $a$  is not a non-negative integer.

We will also use  $\emptyset$  to denote the empty set and  $\mathbf{0}$  to denote the zero vector. For  $J \subseteq I$ ,  $\boldsymbol{\alpha}|_J$  denotes the vector obtained by projecting  $\boldsymbol{\alpha}$  onto  $J$ .

It was shown in [3, Theorem 5] that

$$\hat{M}_g^{(n)}(x, I, \boldsymbol{\alpha}) \approx \hat{\phi}_g^{(n)}(I, \boldsymbol{\alpha})(1 - 12x)^{-(10g+2n+5|I|+2|\boldsymbol{\alpha}|-3)/4}$$

as  $x \rightarrow 1/12$ , where  $\hat{\phi}_g^{(n)}(I, \boldsymbol{\alpha})$  satisfy recursion [3, (4.2)]. With  $t = n + 1$  and noting

$$d_t = \frac{6}{125} \hat{\phi}_0^{(t)}(\emptyset, \mathbf{0}), \quad (t \geq 1)$$

we can rewrite [3, (4.2)] as the following recursion.

$$\begin{aligned} & - \binom{n+1}{n} \hat{\phi}_0^{(1)}(\emptyset, \mathbf{0}) \hat{\phi}_g^{(n)}(I, \boldsymbol{\alpha}) \\ = & \sum_{k=0}^{n-1} \binom{n+1}{k} \hat{\phi}_0^{(n+1-k)}(\emptyset, \mathbf{0}) \hat{\phi}_g^{(k)}(I, \boldsymbol{\alpha}) \\ & + \frac{1}{2} \sum_{j=0}^g \sum_{J \subseteq I} \sum_{k=0}^{n+1} \binom{n+1}{k} \hat{\phi}_j^{(k)}(J, \boldsymbol{\alpha}|_J) \hat{\phi}_{g-j}^{(n+1-k)}(I - J, \boldsymbol{\alpha}|_{I-J}) \\ & \quad (j, J) \neq (0, \emptyset), (g, I) \\ & + \frac{3}{5} \sum_{k=0}^{n+1} \binom{n+1}{k} \hat{\phi}_{g-1}^{(n+1-k)}(I + \{\omega\}, \boldsymbol{\alpha} + (k+1)e_\omega) \\ & + \frac{3}{5} \sum_{i \in I} \frac{(n+1)! \alpha_i!}{(n + \alpha_i + 2)!} \hat{\phi}_g^{(n+\alpha_i+2)}(I - \{i\}, \boldsymbol{\alpha}|_{I-\{i\}}) \end{aligned} \tag{9}$$

with the initial values

$$\hat{\phi}_0^{(n)}(\emptyset, \mathbf{0}) = 5\sqrt{6} \left( \frac{-25}{18} \right)^n \binom{1/2}{n-1} n!. \tag{10}$$

Also

$$t_g = \frac{1}{\Gamma((5g-3)/2)} \left( \frac{6}{25} \sum_{j=1}^{g-1} \hat{\phi}_j^{(0)}(\emptyset, \mathbf{0}) \hat{\phi}_{g-j}^{(0)}(\emptyset, \mathbf{0}) + \frac{36}{125} \hat{\phi}_{g-1}^{(0)}(\{\omega\}, e_\omega) \right). \tag{11}$$

In the above (and the following)  $e_\omega$  denotes the unit vector with a 1 in the  $\omega$ th component (We note that in [3],  $\omega \rightarrow 1$  was used for this purpose). We also note that the above recursion can be used, in the lexicographic order of  $(g, |I|, |\boldsymbol{\alpha}|, n)$ , to compute  $\hat{\phi}_g^{(n)}(I, \boldsymbol{\alpha})$ .

We now turn to the bivariate version of the above recursions.

Let  $\hat{M}_g(u, v, y, I)$  be the bivariate analogy to  $\hat{M}(x, y, I)$  with  $u$  marking the number of faces and  $v$  marking the number of vertices. Define

$$A(u, v, y) = 1 - y + uy^2 + 2u^{-1}y^2(y-1)\hat{M}_0(u, v, y, \emptyset), \tag{12}$$

$$\begin{aligned} B(u, v, y) = & ((1-p)^2(p^2 + 4q^2) - 4q(1-p)^3)y^4 \\ & + 2(4q(1-p)^2 - (1-p)(p + 4q^2))y^3 \\ & + (1 + 4q^2 + (1-p)(2p - 4q))y^2 - 2y + 1, \end{aligned} \tag{13}$$

where

$$u = p(1 - p - 2q), \quad v = q(1 - 2p - q).$$

It was shown in [4] that  $\hat{M}_0(u, v, y, \emptyset)$  satisfies  $A^2 = B$ , and for  $g > 0$ ,  $\hat{M}_g(u, v, y, I)$  is determined by the following recursion

$$\begin{aligned} & A(u, v, y)\hat{M}_g(u, v, y, I) \\ = & -\frac{y^2(y-1)}{u} \sum_{\substack{j=0 \\ (j,J) \neq (0,\emptyset), (g,I)}}^g \sum_{J \subseteq I} \hat{M}_j(u, v, y, J)\hat{M}_{g-j}(u, v, y, I - J) \\ & -\frac{y^3(y-1)}{u} \frac{\partial}{\partial z_w} \hat{M}_{g-1}(u, v, y, I + \{\omega\}) \Big|_{z_w=y} \\ & -uy(y-1) \sum_{i \in I} \frac{z_i}{z_i - y} \left[ z_i \hat{M}_g(u, v, z_i, I - \{i\}) - y \hat{M}_g(u, v, y, I - \{i\}) \right] \\ & +uy\hat{M}_g(u, v, 1, I). \end{aligned} \tag{14}$$

We note that this is the orientable analogy to [4, (4.1)].

Let the parameters  $r$  and  $s$  be related to  $p$  and  $q$  by

$$p = \frac{r}{2(1+r+s)}, \quad q = \frac{s}{2(1+r+s)}.$$

Then

$$u = \frac{r(2+r)}{4(1+r+s)^2}, \quad v = \frac{s(2+s)}{4(1+r+s)^2}.$$

Let

$$y_0 = \frac{2(1+r+r^2)}{2+2r+r^2}, \tag{15}$$

$u_0$  be as defined in (6), and

$$B^{(n)} = \frac{\partial^n B(u, v, y)}{\partial y^n} \Big|_{y=1/(1-p)}.$$

It follows from [4, (2.4)] and the expressions for  $B^{(n)}$ ,  $n = 2, 3$ , on page 328 of [4] that

$$\begin{aligned} B^{(0)} &= B^{(1)} = 0, \\ B^{(2)} &= \frac{2(1-rs)}{(1+r+s)^2} = c_2(1-u/u_0)^{1/2} + O(1-u/u_0), \\ B^{(3)} &= -12(1-p)(p(1-2p) + 4q(1-p-q)) = -c_3 + O((1-u/u_0)^{1/2}), \end{aligned}$$

as  $u \rightarrow u_0$ , where

$$c_2 = \frac{2r^2}{(1+r+r^2)^2} \sqrt{3(2+r)(1+r)}, \quad c_3 = \frac{3(1+r)(2+2r+r^2)^2}{(1+r+r^2)^3}. \tag{16}$$

The following results were implicitly used in [4]. For the readers who are not familiar with [3, 4], we briefly outline how they are derived from (14). As in the univariate case, we define

$$\hat{M}_g^{(n)}(u, v, I, \boldsymbol{\alpha}) = \frac{\partial^{n+|\boldsymbol{\alpha}|}}{\partial y^n \prod_{i \in I} \partial z_i^{\alpha_i}} \hat{M}_g(u, v, y, I) \Big|_{y=z_i=1/(1-p)}.$$

Using the above singular expansions of  $B^{(2)}$  and  $B^{(3)}$ , and the same argument used in the proof of [3, Lemma 2], we obtain

$$\hat{M}_0^{(n)}(u, v, \emptyset, \mathbf{0}) \approx \frac{3c_2 u_0}{2c_3 y_0^2 (y_0 - 1)} \sqrt{\frac{c_2}{2}} \left(-\frac{c_3}{3c_2}\right)^n \binom{1/2}{n-1} n! (1 - u/u_0)^{-(2n-3)/4}, \quad (17)$$

where the factor  $\frac{u_0}{2y_0^2(y_0 - 1)}$  comes from the coefficient of  $\hat{M}_0(u, v, y, \emptyset)$  in (12).

Applying

$$\frac{\partial^{n+1+|\boldsymbol{\alpha}|}}{\partial y^{n+1} \prod_{i \in I} \partial z_i^{\alpha_i}} \Big|_{y=z_i=1/(1-p)}$$

to both sides of (14), we obtain (by induction on the lexicographic order of  $(g, |I|, |\boldsymbol{\alpha}|, n)$ ),

$$\hat{M}_g^{(n)}(u, v, I, \boldsymbol{\alpha}) \approx \hat{M}_g^{(n)}(I, \boldsymbol{\alpha}) (1 - u/u_0)^{-(10g+2n+5|I|+2|\boldsymbol{\alpha}|-3)/4}$$

as  $u \rightarrow u_0$ , where  $\hat{M}_g^{(n)}(I, \boldsymbol{\alpha})$  satisfy the following recursion:

$$\begin{aligned} & -\binom{n+1}{n} \hat{M}_0^{(1)}(\emptyset, \mathbf{0}) \hat{M}_g^{(n)}(I, \boldsymbol{\alpha}) \\ = & \sum_{k=0}^{n-1} \binom{n+1}{k} \hat{M}_0^{(n+1-k)}(\emptyset, \mathbf{0}) \hat{M}_g^{(k)}(I, \boldsymbol{\alpha}) \\ & + \frac{1}{2} \sum_{j=0}^g \sum_{J \subseteq I} \sum_{k=0}^{n+1} \binom{n+1}{k} \hat{M}_j^{(k)}(J, \boldsymbol{\alpha}|_J) \hat{M}_{g-j}^{(n+1-k)}(I - J, \boldsymbol{\alpha}|_{I-J}) \\ & \quad (j, J) \neq (0, \emptyset), (g, I) \\ & + \frac{y_0}{2} \sum_{k=0}^{n+1} \binom{n+1}{k} \hat{M}_{g-1}^{(n+1-k)}(I + \{\omega\}, \boldsymbol{\alpha} + (k+1)e_\omega) \\ & + \frac{u_0^2 y_0}{2} \sum_{i \in I} \frac{(n+1)! \alpha_i!}{(n + \alpha_i + 2)!} \hat{M}_g^{(n+\alpha_i+2)}(I - \{i\}, \boldsymbol{\alpha}|_{I-\{i\}}). \end{aligned} \quad (18)$$

Define

$$\beta_0 = \frac{u_0 c_2 \sqrt{3c_2}}{20c_3 y_0^2 (y_0 - 1)}, \quad \beta_1 = \frac{6c_3}{25c_2}, \quad \beta_2 = \frac{5u_0 y_0 \beta_1}{6\beta_0}, \quad \beta_3 = u_0 \beta_2. \quad (19)$$

Then it is not difficult to check that recursions (9) and (18) are equivalent under the transformation

$$\hat{M}_g^{(n)}(I, \boldsymbol{\alpha}) = \beta_0 \beta_1^{n+|\boldsymbol{\alpha}|} \beta_2^{2g} \beta_3^{|I|} \hat{\phi}_g^{(n)}(I, \boldsymbol{\alpha}).$$



Their initial values (10) and (17) are also equivalent under this transformation. Thus we have, for all  $g, n, I, \boldsymbol{\alpha}$ , that

$$\hat{M}_g^{(n)}(I, \boldsymbol{\alpha}) = \beta_0 \beta_1^{n+|\boldsymbol{\alpha}|} \beta_2^{2g} \beta_3^{|\boldsymbol{I}|} \hat{\phi}_g^{(n)}(I, \boldsymbol{\alpha}). \quad (20)$$

Setting  $y = \frac{1}{1-p}$  and  $I = \emptyset$  in (14), we obtain

$$\begin{aligned} \hat{M}_g(u, v_0, 1, \emptyset) &\approx \left( \frac{y_0(y_0 - 1)}{u_0^2} \sum_{j=1}^{g-1} \hat{M}_j^{(0)}(\emptyset, \mathbf{0}) \hat{M}_{g-j}^{(0)}(\emptyset, \mathbf{0}) \right. \\ &\quad \left. + \frac{y_0^2(y_0 - 1)}{u_0^2} \hat{M}_{g-1}^{(0)}(\{\omega\}, e_\omega) \right) (1 - u/u_0)^{-(5g-3)/2}, \end{aligned}$$

as  $u \rightarrow u_0$ .

Using the Flajolet-Odlyzko “transfer theorem” [15, Corollary VI.1], (11) and (20), we obtain

$$\begin{aligned} [u^i] \hat{M}_g(u, v, 1, \emptyset) &\sim \frac{1}{\Gamma((5g-3)/2)} \left( \frac{y_0(y_0 - 1)}{u_0^2} \sum_{j=1}^{g-1} \hat{M}_j^{(0)}(\emptyset, \mathbf{0}) \hat{M}_{g-j}^{(0)}(\emptyset, \mathbf{0}) \right. \\ &\quad \left. + \frac{y_0^2(y_0 - 1)}{u_0^2} \hat{M}_{g-1}^{(0)}(\{w\}, e_w) \right) i^{5(g-1)/2} u_0^{-i} \\ &= \frac{25y_0(y_0 - 1)}{6u_0^2} \beta_0^2 \beta_2^{2g} t_g i^{5(g-1)/2} u_0^{-i}, \end{aligned} \quad (21)$$

as  $i \rightarrow \infty$  uniformly for  $r$  in any closed subinterval of  $(0, \infty)$ .

As indicated in [4], the local limit theorem [10] gives

$$T_g(i, j) = [u^i v^j] \hat{M}_g(u, v, 1, \emptyset) \sim \frac{25y_0(y_0 - 1)}{6u_0^2 \sigma \sqrt{i 2\pi}} \beta_0^2 \beta_2^{2g} t_g i^{5(g-1)/2} u_0^{-i} v_0^{-j},$$

with [4, Lemma 3]

$$\frac{j}{i} = \frac{1 + 2r}{r^2(2 + r)}, \quad \sigma^2 = \frac{(1 + 2r)(1 + r + r^2)(4 + 7r + 4r^2)}{6r^4(1 + r)(2 + r)^2}. \quad (22)$$

This gives the first asymptotic expression in (5) with

$$t_g(r) = \frac{25y_0(y_0 - 1)}{6u_0^2 \sigma \sqrt{2\pi}} \beta_0^2 \beta_2^{2g} \left( \frac{r^2(2 + r)}{1 + 2r} \right)^{(5g-6)/4} t_g. \quad (23)$$

Now (4) follows from (6), (15), (19), (22), and (23). Using  $t_0 = 2/\sqrt{\pi}$  and  $t_1 = 1/24$  [3], we can verify that our expressions for  $t_0(r)$  and  $t_1(r)$  agree with those given in [4, Theorem 1].

### 3 Connection between $p_g(r)$ and $p_g$

In this section, we provide the proof to Theorem 1 for non-orientable surfaces. Since the argument is essentially the same as the one used in the previous section for orientable surfaces, we will just outline where the minor differences are.

In analogy to the orientable case in section 2, let  $M_g(x, y, I)$  ( $M_g(u, v, y, I)$ ) be the generating function for rooted maps with respect edges (faces and vertices) on a surface (orientable or non-orientable) of Euler characteristic  $2 - 2g$ . Hence the surface is either orientable of genus  $g$ , or non-orientable with  $2g$  cross-caps. Then

$$T_g(n) + P_g(n) = [x^n]M_g(x, 1, \emptyset), \quad T_g(i, j) + P_g(i, j) = [u^i v^j]M_g(u, v, 1, \emptyset).$$

It is known [3, (3.6)] that

$$\begin{aligned} t_g + p_g &= \frac{1}{\Gamma((5g-3)/2)} \left( \frac{6}{25} \sum_{j=1/2}^{g-1/2} \phi_j^{(0)}(\emptyset, \mathbf{0}) \phi_{g-j}^{(0)}(\emptyset, \mathbf{0}) \right. \\ &\quad \left. + \frac{72}{125} \phi_{g-1}^{(0)}(\{\omega\}, e_\omega) + \frac{36}{125} \phi_{g-1/2}^{(1)}(\emptyset, \mathbf{0}) \right), \end{aligned} \quad (24)$$

where the constants  $\phi_g^{(k)}(I, \alpha)$  satisfy the following recursion (noting the remark before (10)).

$$\begin{aligned} & - \binom{n+1}{n} \phi_0^{(1)}(\emptyset, \mathbf{0}) \phi_g^{(n)}(I, \alpha) \\ = & \sum_{k=0}^{n-1} \binom{n+1}{k} \phi_0^{(n+1-k)}(\emptyset, \mathbf{0}) \phi_g^{(k)}(I, \alpha) \\ & + \frac{1}{2} \sum_{j=0/2}^g \sum_{J \subseteq I} \sum_{k=0}^{n+1} \binom{n+1}{k} \phi_j^{(k)}(J, \alpha|_J) \phi_{g-j}^{(n+1-k)}(I - J, \alpha|_{I-J}) \\ & \quad (j, J) \neq (0, \emptyset), (g, I) \\ & + \frac{6}{5} \sum_{k=0}^{n+1} \binom{n+1}{k} \phi_{g-1}^{(n+1-k)}(I + \{\omega\}, \alpha + (k+1)e_\omega) \\ & + \frac{3}{5} \phi_{g-1/2}^{(n+2)}(I, \alpha) \\ & + \frac{3}{5} \sum_{i \in I} \frac{(n+1)! \alpha_i!}{(n + \alpha_i + 2)!} \phi_g^{(n+\alpha_i+2)}(I - \{i\}, \alpha|_{I-\{i\}}) \end{aligned} \quad (25)$$

with the initial values given by

$$\phi_0^{(n)}(\emptyset, \mathbf{0}) = \hat{\phi}_0^{(0)}(\emptyset, \mathbf{0}). \quad (\text{as in (10)})$$

We note, in here and below, the summation for  $j$  from  $0/2$  indicates that  $j$  is over all the half integers in the specified range.

As in the previous section, we obtain from [4, (4.1)] that

$$\begin{aligned}
& -\binom{n+1}{n} M_0^{(1)}(\emptyset, \mathbf{0}) M_g^{(n)}(I, \boldsymbol{\alpha}) \\
= & \sum_{k=0}^{n-1} \binom{n+1}{k} M_0^{(n+1-k)}(\emptyset, \mathbf{0}) M_g^{(k)}(I, \boldsymbol{\alpha}) \\
& + \frac{1}{2} \sum_{\substack{j=0/2 \\ (j,J) \neq (0,\emptyset), (g,I)}}^g \sum_{J \subseteq I} \sum_{k=0}^{n+1} \binom{n+1}{k} M_j^{(k)}(J, \boldsymbol{\alpha}|_J) M_{g-j}^{(n+1-k)}(I-J, \boldsymbol{\alpha}|_{I-J}) \\
& + y_0 \sum_{k=0}^{n+1} \binom{n+1}{k} M_{g-1}^{(n+1-k)}(I + \{\omega\}, \boldsymbol{\alpha} + (k+1)e_\omega) \\
& + \frac{u_0 y_0}{2} M_{g-1/2}^{(n+2)}(I, \boldsymbol{\alpha}) \\
& + \frac{u_0^2 y_0}{2} \sum_{i \in I} \frac{(n+1)! \alpha_i!}{(n + \alpha_i + 2)!} M_g^{(n+\alpha_i+2)}(I - \{i\}, \boldsymbol{\alpha}|_{I-\{i\}}).
\end{aligned} \tag{26}$$

Let  $\beta_0, \beta_1, \beta_2, \beta_3$  be as defined in (19), it is easy to verify that

$$\beta_0 \beta_1^{n+|\boldsymbol{\alpha}|} \beta_2^{2g} \beta_3^{|I|} \phi_g^{(n)}(I, \boldsymbol{\alpha})$$

satisfy (26), and hence

$$M_g^{(n)}(I, \boldsymbol{\alpha}) = \beta_0 \beta_1^{n+|\boldsymbol{\alpha}|} \beta_2^{2g} \beta_3^{|I|} \phi_g^{(n)}(I, \boldsymbol{\alpha}).$$

As in the previous section, this implies that

$$\begin{aligned}
[u^i] M_g(u, v, 1, \emptyset) & \sim \frac{1}{\Gamma((5g-3)/2)} \left( \frac{y_0(y_0-1)}{u_0^2} \sum_{j=1/2}^{g-1/2} M_j^{(0)}(\emptyset, \mathbf{0}) M_{g-j}^{(0)}(\emptyset, \mathbf{0}) \right. \\
& + \frac{2y_0^2(y_0-1)}{u_0^2} M_{g-1}^{(0)}(\{\omega\}, e_\omega) \\
& \left. + \frac{y_0^2(y_0-1)}{u_0} M_{g-1/2}^{(1)}(\emptyset, \mathbf{0}) \right) i^{5(g-1)/2} u_0^{-i} \\
& = \frac{25y_0(y_0-1)}{6u_0^2} \beta_0^2 \beta_2^{2g} (t_g + p_g) i^{5(g-1)/2} u_0^{-i}.
\end{aligned} \tag{27}$$

Again, as indicated in [4], the local limit theorem gives

$$T_g(i, j) + P_g(i, j) \sim \frac{25y_0(y_0-1)}{6u_0^2 \sigma \sqrt{i} 2\pi} \beta_0^2 \beta_2^{2g} (t_g + p_g) i^{5(g-1)/2} u_0^{-i} v_0^{-j},$$

and hence

$$t_g(r) + p_g(r) = \frac{25y_0(y_0 - 1)}{6u_0^2\sigma\sqrt{2\pi}}\beta_0^2\beta_2^{2g} \left( \frac{r^2(2+r)}{1+2r} \right)^{(5g-6)/4} (t_g + p_g).$$

This together with (23) gives

$$p_g(r) = \frac{25y_0(y_0 - 1)}{6u_0^2\sigma\sqrt{2\pi}}\beta_0^2\beta_2^{2g} \left( \frac{r^2(2+r)}{1+2r} \right)^{(5g-6)/4} p_g. \quad (28)$$

Now (5) follows from (6), (15), (19), (22), and (28). This completes the proof of Theorem 1. Using  $p_{1/2} = -2\sqrt{6}/\Gamma(-1/4)$ , we can verify that our expression for  $p_{1/2}(r)$  agrees with that given in [4, Theorem 1].

## 4 Concluding remarks

In this paper, we derived a simple expression for the coefficients  $t_g(r)$  ( $p_g(r)$ ) in the asymptotic formula for the number of rooted maps on an orientable (non-orientable) surface with Euler characteristic  $2 - 2g$ , with respect to faces and vertices. As shown in Theorem 1,  $t_g(r) = c(r)[d(r)]^g t_g$  for some simple algebraic functions  $c(r)$  and  $d(r)$ . Since  $t_g$  can be efficiently computed using (3), so can  $t_g(r)$ . Furthermore, the asymptotic expression for  $t_g$  leads to an asymptotic expression for  $t_g(r)$ . Also if the conjecture (4) of Garoufalidis and Mariño is true, then both  $p_g$  and  $p_g(r)$  can be efficiently computed. This implies that the coefficients in the asymptotic formulas for many families of maps and graphs can be computed efficiently.

We also mention that some results are known for computing the exact values of  $T_g(n)$ ,  $P_g(n)$ ,  $T_g(i, j)$  and  $P_g(i, j)$ . For example, Arquès and Giorgetti [1, 2] showed

$$\sum_{i,j \geq 1} T_g(i, j) u^i v^j = \frac{pq(1-p-q)\hat{Q}_g(p, q)}{[(1-2p-2q)^2 - 4pq]^{5g-3}},$$

$$\sum_{i,j \geq 1} (T_g(i, j) + P_g(i, j)) u^i v^j = \frac{Q_g(p, q, t)}{[(1-2p-2q)^2 - 4pq]^{5g-3}},$$

where  $\hat{Q}_g(p, q)$  is a polynomial in  $p, q$  with total degree at most  $6g - 3$ , and  $Q_g(p, q, t)$  is a polynomial in  $p, q$ , and  $t = \sqrt{(1-2p-2q)^2 - 4pq}$  with total degree at most  $6g - 6$ .

Since the above results were obtained using complicated recursions like (14), so far there is no efficient way known for computing  $\hat{Q}_g(p, q)$  and  $Q_g(p, q, t)$ . In view of (2), there might be simple recursions for  $T_g(n)$  and  $P_g(n)$ , or even for  $T_g(i, j)$  and  $P_g(i, j)$ . Indeed, it will be very interesting to find such simple recursions.

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