Upper Bounds on the Spanning Ratio of Constrained Theta-Graphs*

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Abstract. We present tight upper and lower bounds on the spanning ratio of a large family of constrained θ-graphs. We show that constrained θ-graphs with $4k+2$ ($k \geq 1$ and integer) cones have a tight spanning ratio of $1 + 2\sin(\theta/2)$, where θ is $2\pi/(4k+2)$. We also present improved upper bounds on the spanning ratio of the other families of constrained θ-graphs.

1 Introduction

A geometric graph G is a graph whose vertices are points in the plane and whose edges are line segments between pairs of points. Every edge is weighted by the Euclidean distance between its endpoints. The distance between two vertices u and v in G, denoted by $d_G(u, v)$, is defined as the sum of the weights of the edges along the shortest path between u and v in G. A subgraph H of G is a t-spanner of G (for $t \geq 1$) if for each pair of vertices u and v, $d_H(u, v) \leq t \cdot d_G(u, v)$. The smallest value t for which H is a t-spanner is the *spanning ratio* or *stretch factor*. The graph G is referred to as the *underlying graph* of H. The spanning properties of various geometric graphs have been studied extensively in the literature (see [4,9] for a comprehensive overview of the topic). We look at a specific type of geometric spanner: θ-graphs.

Introduced independently by Clarkson [6] and Keil [8], θ-graphs partition the plane around each vertex into m disjoint cones, each having aperture $\theta = 2\pi/m$. The θ_m-graph is constructed by, for each cone of each vertex u, connecting u to the vertex v whose projection along the bisector of the cone is closest. Ruppert and Seidel [10] showed that the spanning ratio of these graphs is at most $1/(1 - 2\sin(\theta/2))$, when $\theta < \pi/3$, i.e. there are at least seven cones. Recent results include a tight spanning ratio of $1 + 2\sin(\theta/2)$ for θ-graphs with $4k + 2$ cones [1], where $k \geq 1$ and integer, and improved upper bounds for the other three families of θ-graphs [5].

Most of the research, however, has focused on constructing spanners where the underlying graph is the complete Euclidean geometric graph. We study this problem in a more general setting with the introduction of line segment *constraints*. Specifically, let P be a set of points in the plane and let S be a set

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of line segments between two vertices in P, called *constraints*. The set of constraints is planar, i.e. no two constraints intersect properly. Two vertices u and v can see each other if and only if either the line segment uv does not properly intersect any constraint or uv is itself a constraint. If two vertices u and v can see each other, the line segment uv is a *visibility edge*. The *visibility graph* of P with respect to a set of constraints S, denoted $Vis(P, S)$, has P as vertex set and all visibility edges as edge set. In other words, it is the complete graph on P minus all edges that properly intersect one or more constraints in S.

This setting has been studied extensively within the context of motion planning amid obstacles. Clarkson [6] was one of the first to study this problem and showed how to construct a linear-sized $(1+\epsilon)$-spanner of $Vis(P, S)$. Subsequently, Das [7] showed how to construct a spanner of $Vis(P, S)$ with constant spanning ratio and constant degree. The Constrained Delaunay Triangulation was shown to be a 2.42-spanner of $Vis(P, S)$ [3]. Recently, it was also shown that the constrained θ_6-graph is a 2-spanner of $Vis(P, S)$ [2]. In this paper, we generalize the recent results on unconstrained θ-graphs to the constrained setting. There are two main obstacles that differentiate this work from previous results. First, the main difficulty with the constrained setting is that induction cannot be applied directly, as the destination need not be visible from the vertex closest to the source (see Figure 5, where w is not visible from v_0, the vertex closest to u). Second, when the graph does not have $4k + 2$ cones, the cones do not line up as nicely as in [2], making it more difficult to apply induction.

In this paper, we overcome these two difficulties and show that constrained θ-graphs with $4k+2$ cones have a spanning ratio of at most $1+2\sin(\theta/2)$, where θ is $2\pi/(4k + 2)$. Since the lower bounds of the unconstrained θ-graphs carry over to the constrained setting, this shows that this spanning ratio is tight. We also show that constrained θ-graphs with $4k + 4$ cones have a spanning ratio of at most $1 + 2\sin(\theta/2)/(\cos(\theta/2) - \sin(\theta/2))$, where θ is $2\pi/(4k + 4)$. Finally, we show that constrained θ-graphs with $4k+3$ or $4k+5$ cones have a spanning ratio of at most $\cos(\theta/4)/(\cos(\theta/2)-\sin(3\theta/4))$, where θ is $2\pi/(4k+3)$ or $2\pi/(4k+5)$.

2 Preliminaries

We define a *cone* C to be the region in the plane between two rays originating from a vertex referred to as the apex of the cone. When constructing a (constrained) $\theta_{(4k+x)}$-graph, for each vertex u consider the rays originating from u with the angle between consecutive rays being $\theta = 2\pi/(4k + x)$, where $k \geq 1$ and integer and $x \in \{2, 3, 4, 5\}$. Each pair of consecutive rays defines a cone. The cones are oriented such that the bisector of some cone coincides with the vertical halfline through u that lies above u. Let this cone be C_0 of u and number the cones in clockwise order around u. The cones around the other vertices have the same orientation as the ones around u. We write C_i^u to indicate the i-th cone of a vertex u. For ease of exposition, we only consider point sets in general position: no two points lie on a line parallel to one of the rays that define the cones, no two points lie on a line perpendicular to the bisector of a cone, and no three points are collinear.

Let vertex u be an endpoint of a constraint c and let the other endpoint v lie in cone C_i^u. The lines through all such constraints c split C_i^u into several *subcones*. We use $C_{i,j}^u$ to denote the j-th subcone of C_i^u. When a constraint $c = (u, v)$ splits a cone of u into two subcones, we define v to lie in both of these subcones. We consider a cone that is not split to be a single subcone.

We now introduce the constrained $\theta_{(4k+x)}$-graph: for each subcone $C_{i,j}$ of each vertex u, add an edge from u to the closest vertex in that subcone that can see u, where distance is measured along the bisector of the original cone (*not the subcone*). More formally, we add an edge between two vertices u and v if v can see u, $v \in C_{i,j}^u$, and for all points $w \in C_{i,j}^u$ that can see u, $|uv'| \leq |uw'|$, where v' and w' denote the projection of v and w on the bisector of C_i^u and $|xy|$ denotes the length of the line segment between two points x and y. Note that our assumption of general position implies that each vertex adds at most one edge for each of its subcones.

Given a vertex w in the cone C_i of vertex u, we define the *canonical triangle* T_{uw} to be the triangle defined by the borders of C_i^u and the line through w perpendicular to the bisector of C_i^u. Note that subcones do not define canonical triangles. We use m to denote the midpoint of the side of T_{uw} opposing u and α to denote the unsigned angle between uw and um (see Figure 1). Note that for any pair of vertices u and w, there exist two canonical triangles: T_{uw} and T_{wu}. We say that a region is *empty* if it does not contain any vertex of P.

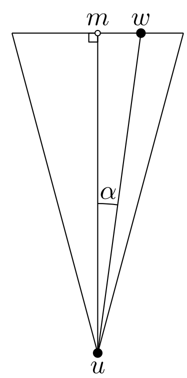

Fig. 1. The canonical triangle T_{uw}

3 Some Useful Lemmas

In this section, we list a number of lemmas that are used when bounding the spanning ratio of the various graphs. Note that these lemmas are not new, as they are already used in [2,5], though some are expanded to work for all four families of constrained θ-graphs. We start with a nice property of visibility graphs from [2].

Lemma 1. *Let u, v, and w be three arbitrary points in the plane such that uw and vw are visibility edges and w is not the endpoint of a constraint intersecting the interior of triangle uvw. Then there exists a convex chain of visibility edges from u to v in triangle uvw, such that the polygon defined by uw, wv and the convex chain is empty and does not contain any constraints.*

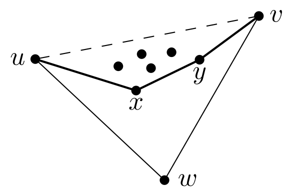

Fig. 2. The convex chain between vertices u and v, where thick lines are visibility edges

Next, we use two lemmas from [5] to bound the length of certain line segments. Note that Lemma 2 is extended such that it also holds for the constrained $\theta_{(4k+2)}$-graph. We use $\angle xyz$ to denote the smaller angle between line segments xy and yz.

Lemma 2. *Let u, v and w be three vertices in the $\theta_{(4k+x)}$-graph, $x \in \{2, 3, 4, 5\}$, such that $w \in C_0^u$ and $v \in T_{uw}$, to the left of uw. Let a be the intersection of the side of T_{uw} opposite u and the left boundary of C_0^v. Let C_i^v denote the cone of v that contains w and let c and d be the upper and lower corner of T_{vw}. If $1 \leq i \leq k-1$, or $i = k$ and $|cw| \leq |dw|$, then* $\max\{|vc| + |cw|, |vd| + |dw|\} \leq |va| + |aw|$ *and* $\max\{|cw|, |dw|\} \leq |aw|$.

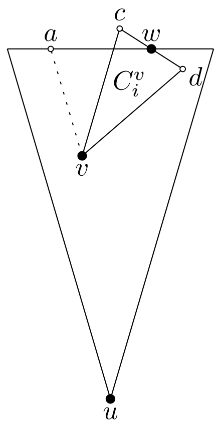

Fig. 3. The situation where we apply Lemma 2

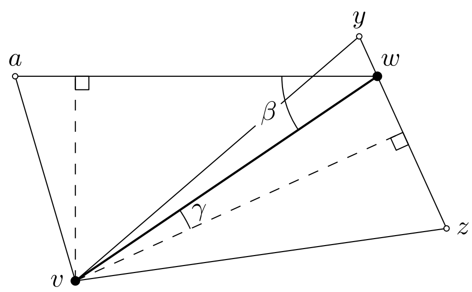

Fig. 4. The situation where we apply Lemma 3

Lemma 3. *Let u, v and w be three vertices in the $\theta_{(4k+x)}$-graph, $x \in \{2, 3, 4, 5\}$, such that $w \in C_0^u$, $v \in T_{uw}$ to the left of uw, and $w \notin C_0^v$. Let a be the intersection of the side of T_{uw} opposite u and the line through v parallel to the left boundary of T_{uw}. Let y and z be the corners of T_{vw} opposite to v. Let $\beta = \angle awv$ and let γ be the unsigned angle between vw and the bisector of T_{vw}. Let $\boldsymbol{c}$ be a positive constant. If $\boldsymbol{c} \geq \frac{\cos\gamma - \sin\beta}{\cos\left(\frac{\theta}{2} - \beta\right) - \sin\left(\frac{\theta}{2} + \gamma\right)}$, then $|vp| + \boldsymbol{c} \cdot |pw| \leq |va| + \boldsymbol{c} \cdot |aw|$, where p is y if $|yw| \geq |zw|$ and z if $|yw| < |zw|$.*

4 Constrained $\theta_{(4k+2)}$-Graph

In this section we prove that the constrained $\theta_{(4k+2)}$-graph has spanning ratio at most $1 + 2 \cdot \sin(\theta/2)$. Since this is also a lower bound [1], this proves that this spanning ratio is tight.

Theorem 1. *Let u and w be two vertices in the plane such that u can see w. Let m be the midpoint of the side of T_{uw} opposing u and let α be the unsigned angle between uw and um. There exists a path connecting u and w in the constrained $\theta_{(4k+2)}$-graph of length at most*

$$\left(\left(\frac{1+\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}\right)\cdot\cos\alpha+\sin\alpha\right)\cdot|uw|.$$

Proof. We assume without loss of generality that $w \in C_0^u$. We prove the theorem by induction on the area of T_{uw}. Formally, we perform induction on the rank, when ordered by area, of the triangles T_{xy} for all pairs of vertices x and y that can see each other. Let a and b be the upper left and right corner of T_{uw}, and let A and B be the triangles uaw and ubw (see Figure 5).

Our inductive hypothesis is the following, where $\delta(u,w)$ denotes the length of the shortest path from u to w in the constrained $\theta_{(4k+2)}$-graph:

- If A is empty, then $\delta(u,w) \leq |ub| + |bw|$.
- If B is empty, then $\delta(u,w) \leq |ua| + |aw|$.
- If neither A nor B is empty, then $\delta(u,w) \leq \max\{|ua| + |aw|, |ub| + |bw|\}$.

We first show that this induction hypothesis implies the theorem: $|um| = |uw|\cdot\cos\alpha$, $|mw| = |uw|\cdot\sin\alpha$, $|am| = |bm| = |uw|\cdot\cos\alpha\cdot\tan(\theta/2)$, and $|ua| = |ub| = |uw|\cdot\cos\alpha/\cos(\theta/2)$. Thus the induction hypothesis gives that $\delta(u,w)$ is at most $|uw|\cdot(((1+\sin(\theta/2))/\cos(\theta/2))\cdot\cos\alpha+\sin\alpha)$.

Base case: T_{uw} has rank 1. Since the triangle is a smallest triangle, w is the closest vertex to u in that cone. Hence the edge (u,w) is part of the constrained $\theta_{(4k+2)}$-graph, and $\delta(u,w) = |uw|$. From the triangle inequality, we have $|uw| \leq \min\{|ua| + |aw|, |ub| + |bw|\}$, so the induction hypothesis holds.

Induction step: We assume that the induction hypothesis holds for all pairs of vertices that can see each other and have a canonical triangle whose area is smaller than the area of T_{uw}.

If (u,w) is an edge in the constrained $\theta_{(4k+2)}$-graph, the induction hypothesis follows by the same argument as in the base case. If there is no edge between u and w, let v_0 be the vertex closest to u in the subcone of u that contains w, and let a_0 and b_0 be the upper left and right corner of T_{uv_0} (see Figure 5). By definition, $\delta(u,w) \leq |uv_0| + \delta(v_0,w)$, and by the triangle inequality, $|uv_0| \leq \min\{|ua_0|+|a_0v_0|, |ub_0|+|b_0v_0|\}$. We assume without loss of generality that v_0 lies to the left of uw, which means that A is not empty.

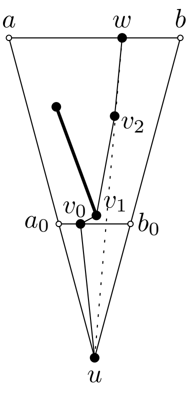

Fig. 5. A convex chain from v_0 to w

Since uw and uv_0 are visibility edges, by applying Lemma 1 to triangle v_0uw, a convex chain $v_0, ..., v_l = w$ of visibility edges connecting v_0 and w exists (see Figure 5). Note that, since v_0 is the closest visible vertex to u, every vertex along the convex chain lies above the horizontal line through v_0.

We now look at two consecutive vertices v_{j-1} and v_j along the convex chain. There are four types of configurations (see Figure 6): (i) $v_j \in C_k^{v_{j-1}}$, (ii) $v_j \in C_i^{v_{j-1}}$ where $1 \leq i < k$, (iii) $v_j \in C_0^{v_{j-1}}$ and v_j lies to the right of or has the same x-coordinate as v_{j-1}, (iv) $v_j \in C_0^{v_{j-1}}$ and v_j lies to the left of v_{j-1}. By convexity, the direction of $\overrightarrow{v_j v_{j+1}}$ is rotating counterclockwise for increasing j. Thus, these configurations occur in the order Type (i), Type (ii), Type (iii), Type (iv) along the convex chain from v_0 to w. We bound $\delta(v_{j-1}, v_j)$ as follows:

Type (i): If $v_j \in C_k^{v_{j-1}}$, let a_j and b_j be the upper and lower left corner of $T_{v_j v_{j-1}}$ and let $B_j = v_{j-1} b_j v_j$. Note that since $v_j \in C_k^{v_{j-1}}$, a_j is also the intersection of the left boundary of $C_0^{v_{j-1}}$ and the horizontal line through v_j. Triangle B_j lies between the convex chain and uw, so it must be empty. Since v_j can see v_{j-1} and $T_{v_j v_{j-1}}$ has smaller area than T_{uw}, the induction hypothesis gives that $\delta(v_{j-1}, v_j)$ is at most $|v_{j-1} a_j| + |a_j v_j|$.

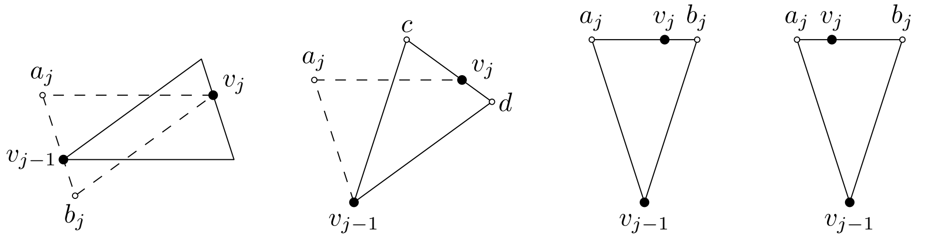

Fig. 6. The four types of configurations

Type (ii): If $v_j \in C_i^{v_{j-1}}$ where $1 \leq i < k$, let c and d be the upper and lower right corner of $T_{v_{j-1} v_j}$. Let a_j be the intersection of the left boundary of $C_0^{v_{j-1}}$ and the horizontal line through v_j. Since v_j can see v_{j-1} and $T_{v_{j-1} v_j}$ has smaller area than T_{uw}, the induction hypothesis gives that $\delta(v_{j-1}, v_j)$ is at most $\max\{|v_{j-1} c| + |c v_j|, |v_{j-1} d| + |d v_j|\}$. Since $v_j \in C_i^{v_{j-1}}$ where $1 \leq i < k$, we can apply Lemma 2 (where v, w, and a from Lemma 2 are v_{j-1}, v_j, and a_j), which gives us that $\max\{|v_{j-1} c| + |c v_j|, |v_{j-1} d| + |d v_j|\} \leq |v_{j-1} a_j| + |a_j v_j|$.

Type (iii): If $v_j \in C_0^{v_{j-1}}$ and v_j lies to the right of or has the same x-coordinate as v_{j-1}, let a_j and b_j be the left and right corner of $T_{v_{j-1} v_j}$ and let $A_j = v_{j-1} a_j v_j$ and $B_j = v_{j-1} b_j v_j$. Since v_j can see v_{j-1} and $T_{v_{j-1} v_j}$ has smaller area than T_{uw}, we can apply the induction hypothesis. Regardless of whether A_j and B_j are empty or not, $\delta(v_{j-1}, v_j)$ is at most $\max\{|v_{j-1} a_j| + |a_j v_j|, |v_{j-1} b_j| + |b_j v_j|\}$. Since v_j lies to the right of or has the same x-coordinate as v_{j-1}, we know that $|v_{j-1} a_j| + |a_j v_j| \geq |v_{j-1} b_j| + |b_j v_j|$, so $\delta(v_{j-1}, v_j)$ is at most $|v_{j-1} a_j| + |a_j v_j|$.

Type (iv): If $v_j \in C_0^{v_{j-1}}$ and v_j lies to the left of v_{j-1}, let a_j and b_j be the left and right corner of $T_{v_{j-1} v_j}$ and let $A_j = v_{j-1} a_j v_j$ and $B_j = v_{j-1} b_j v_j$. Since v_j can see v_{j-1} and $T_{v_{j-1} v_j}$ has smaller area than T_{uw}, we can apply the

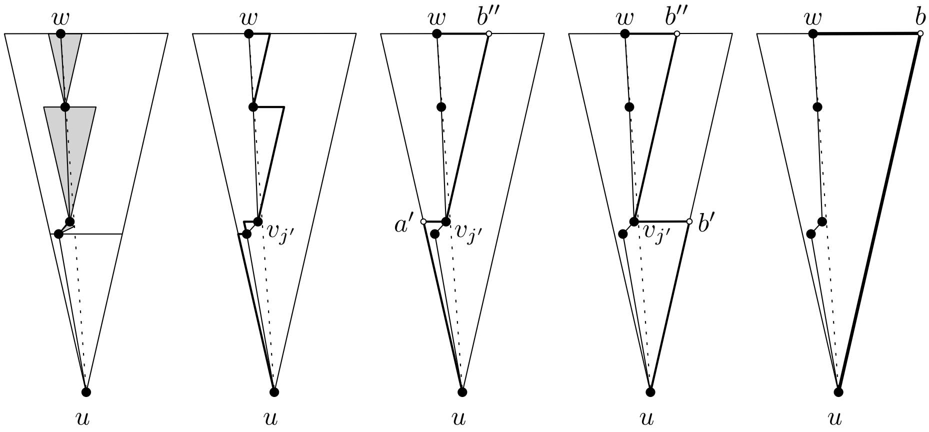

Fig. 7. Visualization of the paths (thick lines) in the inequalities of case (c)

induction hypothesis. Thus, if B_j is empty, $\delta(v_{j-1}, v_j)$ is at most $|v_{j-1}a_j| + |a_j v_j|$ and if B_j is not empty, $\delta(v_{j-1}, v_j)$ is at most $|v_{j-1}b_j| + |b_j v_j|$.

To complete the proof, we consider three cases: (a) $\angle awu \leq \pi/2$, (b) $\angle awu > \pi/2$ and B is empty, (c) $\angle awu > \pi/2$ and B is not empty.

Case (a): If $\angle awu \leq \pi/2$, the convex chain cannot contain any Type (iv) configurations: for Type (iv) configurations to occur, v_j needs to lie to the left of v_{j-1}. However, by construction, v_j lies on or to the right of the line through v_{j-1} and w. Hence, since $\angle awv_{j-1} < \angle awu \leq \pi/2$, v_j lies to the right of or has the same x-coordinate as v_{j-1}. We can now bound $\delta(u, w)$ by using these bounds: $\delta(u, w) \leq |uv_0| + \sum_{j=1}^{l} \delta(v_{j-1}, v_j) \leq |ua_0| + |a_0 v_0| + \sum_{j=1}^{l}(|v_{j-1}a_j| + |a_j v_j|) = |ua| + |aw|$.

Case (b): If $\angle awu > \pi/2$ and B is empty, the convex chain can contain Type (iv) configurations. However, since B is empty and the area between the convex chain and uw is empty (by Lemma 1), all B_j are also empty. Using the computed bounds on the lengths of the paths between the points along the convex chain, we can bound $\delta(u, w)$ as in the previous case.

Case (c): If $\angle awu > \pi/2$ and B is not empty, the convex chain can contain Type (iv) configurations and since B is not empty, the triangles B_j need not be empty. Recall that v_0 lies in A, hence neither A nor B are empty. Therefore, it suffices to prove that $\delta(u, w) \leq \max\{|ua| + |aw|, |ub| + |bw|\} = |ub| + |bw|$. Let $T_{v_{j'} v_{j'+1}}$ be the first Type (iv) configuration along the convex chain (if it has any), let a' and b' be the upper left and right corner of $T_{uv_{j'}}$, and let b'' be the upper right corner of $T_{v_{j'}w}$. We now have that $\delta(u, w) \leq |uv_0| + \sum_{j=1}^{l} \delta(v_{j-1}, v_j) \leq |ua'| + |a'v_{j'}| + |v_{j'}b''| + |b''w| \leq |ub| + |bw|$ (see Figure 7). □

Since $((1 + \sin(\theta/2))/\cos(\theta/2)) \cdot \cos\alpha + \sin\alpha$ is increasing for $\alpha \in [0, \theta/2]$, for $\theta \leq \pi/3$, it is maximized when $\alpha = \theta/2$, and we obtain the following corollary:

Corollary 1. *The constrained* $\theta_{(4k+2)}$*-graph is a* $\left(1 + 2 \cdot \sin\left(\frac{\theta}{2}\right)\right)$*-spanner of* $Vis(P, S)$.

5 Generic Framework for the Spanning Proof

Next, we modify the spanning proof from the previous section and provide a generic framework for the spanning proof for the other three families of θ-graphs. After providing this framework, we fill in the blanks for the individual families.

Theorem 2. *Let u and w be two vertices in the plane such that u can see w. Let m be the midpoint of the side of T_{uw} opposing u and let α be the unsigned angle between uw and um. There exists a path connecting u and w in the constrained $\theta_{(4k+x)}$-graph of length at most*

$$\left(\frac{\cos \alpha}{\cos\left(\frac{\theta}{2}\right)} + \left(\cos \alpha \cdot \tan\left(\frac{\theta}{2}\right) + \sin \alpha \right) \cdot \boldsymbol{c} \right) \cdot |uw|,$$

where $\boldsymbol{c} \geq 1$ is a constant that depends on $x \in \{3, 4, 5\}$. For the constrained $\theta_{(4k+4)}$-graph, $\boldsymbol{c}$ equals $1/(\cos(\theta/2) - \sin(\theta/2))$ and for the constrained $\theta_{(4k+3)}$-graph and $\theta_{(4k+5)}$-graph, $\boldsymbol{c}$ equals $\cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4))$.

Proof. We prove the theorem by induction on the area of T_{uw}. Formally, we perform induction on the rank, when ordered by area, of the triangles T_{xy} for all pairs of vertices x and y that can see each other. We assume without loss of generality that $w \in C_0^u$. Let a and b be the upper left and right corner of T_{uw} (see Figure 5).

Our inductive hypothesis is the following, where $\delta(u, w)$ denotes the length of the shortest path from u to w in the constrained $\theta_{(4k+x)}$-graph: $\delta(u, w) \leq \max\{|ua| + |aw| \cdot \boldsymbol{c}, |ub| + |bw| \cdot \boldsymbol{c}\}$.

We first show that this induction hypothesis implies the theorem. Basic trigonometry gives us the following equalities: $|um| = |uw| \cdot \cos \alpha$, $|mw| = |uw| \cdot \sin \alpha$, $|am| = |bm| = |uw| \cdot \cos \alpha \cdot \tan(\theta/2)$, and $|ua| = |ub| = |uw| \cdot \cos \alpha / \cos(\theta/2)$. Thus the induction hypothesis gives that $\delta(u, w)$ is at most $|uw| \cdot (\cos \alpha / \cos(\theta/2) + (\cos \alpha \cdot \tan(\theta/2) + \sin \alpha) \cdot \boldsymbol{c})$.

Base case: T_{uw} has rank 1. Since the triangle is a smallest triangle, w is the closest vertex to u in that cone. Hence the edge (u, w) is part of the constrained $\theta_{(4k+x)}$-graph, and $\delta(u, w) = |uw|$. From the triangle inequality and the fact that $\boldsymbol{c} \geq 1$, we have $|uw| \leq \min\{|ua| + |aw| \cdot \boldsymbol{c}, |ub| + |bw| \cdot \boldsymbol{c}\}$, so the induction hypothesis holds.

Induction step: We assume that the induction hypothesis holds for all pairs of vertices that can see each other and have a canonical triangle whose area is smaller than the area of T_{uw}.

If (u, w) is an edge in the constrained $\theta_{(4k+x)}$-graph, the induction hypothesis follows by the same argument as in the base case. If there is no edge between u and w, let v_0 be the vertex closest to u in the subcone of u that contains w, and let a_0 and b_0 be the upper left and right corner of T_{uv_0} (see Figure 5). By definition, $\delta(u, w) \leq |uv_0| + \delta(v_0, w)$, and by the triangle inequality, $|uv_0| \leq \min\{|ua_0| + |a_0v_0|, |ub_0| + |b_0v_0|\}$. We assume without loss of generality that v_0 lies to the left of uw.

Since uw and uv_0 are visibility edges, by applying Lemma 1 to triangle v_0uw, a convex chain $v_0, ..., v_l = w$ of visibility edges connecting v_0 and w exists (see Figure 5). Note that, since v_0 is the closest visible vertex to u, every vertex along the convex chain lies above the horizontal line through v_0.

We now look at two consecutive vertices v_{j-1} and v_j along the convex chain. When $v_j \notin C_0^{v_{j-1}}$, let c and d be the upper and lower right corner of $T_{v_{j-1}v_j}$. We distinguish four types of configurations: (i) $v_j \in C_i^{v_{j-1}}$ where $i > k$, or $i = k$ and $|cw| > |dw|$, (ii) $v_j \in C_i^{v_{j-1}}$ where $1 \leq i \leq k-1$, or $i = k$ and $|cw| \leq |dw|$, (iii) $v_j \in C_0^{v_{j-1}}$ and v_j lies to the right of or has the same x-coordinate as v_{j-1}, (iv) $v_j \in C_0^{v_{j-1}}$ and v_j lies to the left of v_{j-1}. By convexity, the direction of $\overrightarrow{v_j v_{j+1}}$ is rotating counterclockwise for increasing j. Thus, these configurations occur in the order Type (i), Type (ii), Type (iii), Type (iv) along the convex chain from v_0 to w. We bound $\delta(v_{j-1}, v_j)$ as follows:

Type (i): $v_j \in C_i^{v_{j-1}}$ where $i > k$, or $i = k$ and $|cw| > |dw|$. Since v_j can see v_{j-1} and $T_{v_j v_{j-1}}$ has smaller area than T_{uw}, the induction hypothesis gives that $\delta(v_{j-1}, v_j)$ is at most $\max\{|v_{j-1}c| + |cv_j| \cdot \boldsymbol{c}, |v_{j-1}d| + |dv_j| \cdot \boldsymbol{c}\}$.

Let a_j be the intersection of the left boundary of $C_0^{v_{j-1}}$ and the horizontal line through v_j. We aim to show that $\max\{|v_{j-1}c| + |cv_j| \cdot \boldsymbol{c}, |v_{j-1}d| + |dv_j| \cdot \boldsymbol{c}\} \leq |v_{j-1}a_j| + |a_j v_j| \cdot \boldsymbol{c}$. We use Lemma 3 to do this. However, since the precise application of this lemma depends on the family of θ-graphs and determines the value of $\boldsymbol{c}$, this case is discussed in the spanning proofs of the three families.

Type (ii): $v_j \in C_i^{v_{j-1}}$ where $1 \leq i \leq k-1$, or $i = k$ and $|cw| \leq |dw|$. Since v_j can see v_{j-1} and $T_{v_j v_{j-1}}$ has smaller area than T_{uw}, the induction hypothesis gives that $\delta(v_{j-1}, v_j)$ is at most $\max\{|v_{j-1}c| + |cv_j| \cdot \boldsymbol{c}, |v_{j-1}d| + |dv_j| \cdot \boldsymbol{c}\}$.

Let a_j be the intersection of the left boundary of $C_0^{v_{j-1}}$ and the horizontal line through v_j. Since $v_j \in C_i^{v_{j-1}}$ where $1 \leq i \leq k-1$, or $i = k$ and $|cw| \leq |dw|$, we can apply Lemma 2 in this case (where v, w, and a from Lemma 2 are v_{j-1}, v_j, and a_j) and we get that $\max\{|v_{j-1}c| + |cv_j|, |v_{j-1}d| + |dv_j|\} \leq |v_{j-1}a_j| + |a_j v_j|$ and $\max\{|cv_j|, |dv_j|\} \leq |a_j v_j|$. Since $\boldsymbol{c} \geq 1$, this implies that $\max\{|v_{j-1}c| + |cv_j| \cdot \boldsymbol{c}, |v_{j-1}d| + |dv_j| \cdot \boldsymbol{c}\} \leq |v_{j-1}a_j| + |a_j v_j| \cdot \boldsymbol{c}$.

Type (iii): If $v_j \in C_0^{v_{j-1}}$ and v_j lies to the right of or has the same x-coordinate as v_{j-1}, let a_j and b_j be the left and right corner of $T_{v_{j-1}v_j}$. Since v_j can see v_{j-1} and $T_{v_{j-1}v_j}$ has smaller area than T_{uw}, we can apply the induction hypothesis. Thus, since v_j lies to the right of or has the same x-coordinate as v_{j-1}, $\delta(v_{j-1}, v_j)$ is at most $|v_{j-1}a_j| + |a_j v_j| \cdot \boldsymbol{c}$.

Type (iv): If $v_j \in C_0^{v_{j-1}}$ and v_j lies to the left of v_{j-1}, let a_j and b_j be the left and right corner of $T_{v_{j-1}v_j}$. Since v_j can see v_{j-1} and $T_{v_{j-1}v_j}$ has smaller area than T_{uw}, we can apply the induction hypothesis. Thus, since v_j lies to the left of v_{j-1}, $\delta(v_{j-1}, v_j)$ is at most $|v_{j-1}b_j| + |b_j v_j| \cdot \boldsymbol{c}$.

To complete the proof, we consider two cases: (a) $\angle awu \leq \frac{\pi}{2}$, (b) $\angle awu > \frac{\pi}{2}$.

Case (a): We need to prove that $\delta(u, w) \leq \max\{|ua| + |aw|, |ub| + |bw|\} = |ua| + |aw|$. We first show that the convex chain cannot contain any Type (iv) configurations: for Type (iv) configurations to occur, v_j needs to lie to the left of v_{j-1}. However, by construction, v_j lies on or to the right of the line through v_{j-1} and w. Hence, since $\angle awv_{j-1} < \angle awu \leq \pi/2$, v_j lies to the right of v_{j-1}. We can now bound $\delta(u, w)$ by using these bounds: $\delta(u, w) \leq |uv_0| + \sum_{j=1}^{l} \delta(v_{j-1}, v_j) \leq |ua_0| + |a_0 v_0| + \sum_{j=1}^{l}(|v_{j-1}a_j| + |a_j v_j| \cdot \boldsymbol{c}) \leq |ua| + |aw| \cdot \boldsymbol{c}$.

Case (b): If $\angle awu > \pi/2$, the convex chain can contain Type (iv) configurations. We need to prove that $\delta(u, w) \leq \max\{|ua|+|aw|, |ub|+|bw|\} = |ub|+|bw|$. Let $T_{v_{j'} v_{j'+1}}$ be the first Type (iv) configuration along the convex chain (if it has any), let a' and b' be the upper left and right corner of $T_{uv_{j'}}$, and let b'' be the upper right corner of $T_{v_{j'}w}$. We now have that $\delta(u, w) \leq |uv_0| + \sum_{j=1}^{l} \delta(v_{j-1}, v_j) \leq |ua'| + |a'v_{j'}| \cdot \boldsymbol{c} + |v_{j'}b''| + |b''w| \cdot \boldsymbol{c} \leq |ub| + |bw| \cdot \boldsymbol{c}$ (see Figure 7). □

6 The Constrained $\boldsymbol{\theta_{(4k+4)}}$-Graph

In this section we complete the proof of Theorem 2 for the constrained $\theta_{(4k+4)}$-graph.

Theorem 3. *Let u and w be two vertices in the plane such that u can see w. Let m be the midpoint of the side of T_{uw} opposite u and let α be the unsigned angle between uw and um. There exists a path connecting u and w in the constrained $\theta_{(4k+4)}$-graph of length at most*

$$\left(\frac{\cos \alpha}{\cos\left(\frac{\theta}{2}\right)} + \frac{\cos \alpha \cdot \tan\left(\frac{\theta}{2}\right) + \sin \alpha}{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)} \right) \cdot |uw|.$$

Proof. We apply Theorem 2 using $\boldsymbol{c} = 1/(\cos(\theta/2) - \sin(\theta/2))$. The assumptions made in Theorem 2 still apply. It remains to show that for the Type (i) configurations, we have that $\max\{|v_{j-1}c| + |cv_j| \cdot \boldsymbol{c}, |v_{j-1}d| + |dv_j| \cdot \boldsymbol{c}\} \leq |v_{j-1}a_j| + |a_j v_j| \cdot \boldsymbol{c}$, where c and d are the upper and lower right corner of $T_{v_{j-1}v_j}$ and a_j is the intersection of the left boundary of $C_0^{v_{j-1}}$ and the horizontal line through v_j.

We distinguish two cases: (a) $v_j \in C_k^{v_{j-1}}$ and $|cw| > |dw|$, (b) $v_j \in C_{k+1}^{v_{j-1}}$. Let β be $\angle a_j v_j v_{j-1}$ and let γ be the angle between $v_j v_{j-1}$ and the bisector of $T_{v_{j-1}v_j}$.

Case (a): When $v_j \in C_k^{v_{j-1}}$ and $|cw| > |dw|$, the induction hypothesis for $T_{v_{j-1}v_j}$ gives $\delta(v_{j-1}, v_j) \leq |v_{j-1}c| + |cv_j| \cdot \boldsymbol{c}$. We note that $\gamma = \theta - \beta$. Hence Lemma 3 gives that the inequality holds when $\boldsymbol{c} \geq (\cos(\theta - \beta) - \sin\beta)/(\cos(\theta/2 - \beta) - \sin(3\theta/2 - \beta))$. As this function is decreasing in β for $\theta/2 \leq \beta \leq \theta$, it is maximized when β equals $\theta/2$. Hence $\boldsymbol{c}$ needs to be at least $(\cos(\theta/2) - \sin(\theta/2))/(1 - \sin\theta)$, which can be rewritten to $1/(\cos(\theta/2) - \sin(\theta/2))$.

Case (b): When $v_j \in C_{k+1}^{v_{j-1}}$, v_j lies above the bisector of $T_{v_{j-1}v_j}$ and the induction hypothesis for $T_{v_{j-1}v_j}$ gives $\delta(v_{j-1}, v_j) \leq |v_j d| + |dv_{j-1}| \cdot \boldsymbol{c}$. We note that $\gamma = \beta$. Hence Lemma 3 gives that the inequality holds when $\boldsymbol{c} \geq (\cos\beta - \sin\beta)/(\cos(\theta/2 - \beta) - \sin(\theta/2 + \beta))$. As this function is decreasing in β for $0 \leq \beta \leq \theta/2$, it is maximized when β equals 0. Hence $\boldsymbol{c}$ needs to be at least $1/(\cos(\theta/2) - \sin(\theta/2))$. □

Since $\cos\alpha/\cos(\theta/2) + (\cos\alpha \cdot \tan(\theta/2) + \sin\alpha)/(\cos(\theta/2) - \sin(\theta/2))$ is increasing for $\alpha \in [0, \theta/2]$, for $\theta \leq \pi/4$, it is maximized when $\alpha = \theta/2$, and we obtain the following corollary:

Corollary 2. *The constrained $\theta_{(4k+4)}$-graph is a* $\left(1 + \frac{2 \cdot \sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)}\right)$*-spanner of $Vis(P, S)$.*

7 The Constrained $\theta_{(4k+3)}$-Graph and $\theta_{(4k+5)}$-Graph

In this section we complete the proof of Theorem 2 for the constrained $\theta_{(4k+3)}$-graph and $\theta_{(4k+5)}$-graph.

Theorem 4. *Let u and w be two vertices in the plane such that u can see w. Let m be the midpoint of the side of T_{uw} opposite u and let α be the unsigned angle between uw and um. There exists a path connecting u and w in the constrained $\theta_{(4k+3)}$-graph of length at most*

$$\left(\frac{\cos\alpha}{\cos\left(\frac{\theta}{2}\right)} + \frac{\left(\cos\alpha \cdot \tan\left(\frac{\theta}{2}\right) + \sin\alpha\right) \cdot \cos\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{3\theta}{4}\right)}\right) \cdot |uw|.$$

Proof. We apply Theorem 2 using $\boldsymbol{c} = \cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4))$. The assumptions made in Theorem 2 still apply. It remains to show that for the Type (i) configurations, we have that $\max\{|v_{j-1}c| + |cv_j| \cdot \boldsymbol{c}, |v_{j-1}d| + |dv_j| \cdot \boldsymbol{c}\} \leq |v_{j-1}a_j| + |a_j v_j| \cdot \boldsymbol{c}$, where c and d are the upper and lower right corner of $T_{v_{j-1}v_j}$ and a_j is the intersection of the left boundary of $C_0^{v_{j-1}}$ and the horizontal line through v_j.

We distinguish two cases: (a) $v_j \in C_k^{v_{j-1}}$ and $|cw| > |dw|$, (b) $v_j \in C_{k+1}^{v_{j-1}}$. Let β be $\angle a_j v_j v_{j-1}$ and let γ be the angle between $v_j v_{j-1}$ and the bisector of $T_{v_{j-1}v_j}$.

Case (a): When $v_j \in C_k^{v_{j-1}}$ and $|cw| > |dw|$, the induction hypothesis for $T_{v_{j-1}v_j}$ gives $\delta(v_{j-1}, v_j) \leq |v_{j-1}c| + |cv_j| \cdot \boldsymbol{c}$. We note that $\gamma = 3\theta/4 - \beta$. Hence Lemma 3 gives that the inequality holds when $\boldsymbol{c} \geq (\cos(3\theta/4 - \beta) - \sin\beta)/(\cos(\theta/2 - \beta) - \sin(5\theta/4 - \beta))$. As this function is decreasing in β for $\theta/4 \leq \beta \leq 3\theta/4$, it is maximized when β equals $\theta/4$. Hence $\boldsymbol{c}$ needs to be at least $(\cos(\theta/2) - \sin(\theta/4))/(\cos(\theta/4) - \sin\theta)$, which is equal to $\cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4))$.

Case (b): When $v_j \in C_{k+1}^{v_{j-1}}$, v_j lies above the bisector of $T_{v_{j-1}v_j}$ and the induction hypothesis for $T_{v_{j-1}v_j}$ gives $\delta(v_{j-1}, v_j) \leq |v_j d| + |dv_{j-1}| \cdot \boldsymbol{c}$. We note that $\gamma = \theta/4 + \beta$. Hence Lemma 3 gives that the inequality holds when $\boldsymbol{c} \geq (\cos(\theta/4 + \beta) - \sin\beta)/(\cos(\theta/2 - \beta) - \sin(3\theta/4 + \beta))$, which is equal to $\cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4))$. □

Theorem 5. *Let u and w be two vertices in the plane such that u can see w. Let m be the midpoint of the side of T_{uw} opposite u and let α be the unsigned angle between uw and um. There exists a path connecting u and w in the constrained $\theta_{(4k+5)}$-graph of length at most*

$$\left(\frac{\cos\alpha}{\cos\left(\frac{\theta}{2}\right)} + \frac{\left(\cos\alpha \cdot \tan\left(\frac{\theta}{2}\right) + \sin\alpha\right) \cdot \cos\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{3\theta}{4}\right)}\right) \cdot |uw|.$$

Proof. We apply Theorem 2 using $\boldsymbol{c} = \cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4))$. The assumptions made in Theorem 2 still apply. It remains to show that for the Type (i) configurations, we have that $\max\{|v_{j-1}c| + |cv_j| \cdot \boldsymbol{c}, |v_{j-1}d| + |dv_j| \cdot \boldsymbol{c}\} \leq |v_{j-1}a_j| + |a_j v_j| \cdot \boldsymbol{c}$, where c and d are the upper and lower right corner of $T_{v_{j-1}v_j}$ and a_j is the intersection of the left boundary of $C_0^{v_{j-1}}$ and the horizontal line through v_j.

We distinguish two cases: (a) $v_j \in C_k^{v_{j-1}}$ and $|cw| > |dw|$, (b) $v_j \in C_{k+1}^{v_{j-1}}$. Let β be $\angle a_j v_j v_{j-1}$ and let γ be the angle between $v_j v_{j-1}$ and the bisector of $T_{v_{j-1}v_j}$.

Case (a): When $v_j \in C_k^{v_{j-1}}$ and $|cw| > |dw|$, the induction hypothesis for $T_{v_{j-1}v_j}$ gives $\delta(v_{j-1}, v_j) \leq |v_{j-1}c| + |cv_j| \cdot \boldsymbol{c}$. We note that $\gamma = 5\theta/4 - \beta$. Hence Lemma 3 gives that the inequality holds when $\boldsymbol{c} \geq (\cos(5\theta/4 - \beta) - \sin\beta)/(\cos(\theta/2 - \beta) - \sin(5\theta/4 - \beta))$. As this function is decreasing in β for $3\theta/4 \leq \beta \leq 5\theta/4$, it is maximized when β equals $3\theta/4$. Hence $\boldsymbol{c}$ needs to be at least $(\cos(\theta/2) - \sin(3\theta/4))/(\cos(\theta/4) - \sin\theta)$, which is less than $\cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4))$.

Case (b): When $v_j \in C_{k+1}^{v_{j-1}}$, the induction hypothesis for T_{vw} gives $\delta(v_{j-1}, v_j) \leq \max\{|v_{j-1}c| + |cv_j| \cdot \boldsymbol{c}, |v_{j-1}d| + |dv_j| \cdot \boldsymbol{c}\}$. If $\delta(v_{j-1}, v_j) \leq |v_{j-1}c| + |cv_j| \cdot \boldsymbol{c}$, we note that $\gamma = \theta/4 - \beta$. Hence Lemma 3 gives that the inequality holds when $\boldsymbol{c} \geq (\cos(\theta/4 - \beta) - \sin\beta)/(\cos(\theta/2 - \beta) - \sin(3\theta/4 - \beta))$. As this function is decreasing in β for $0 \leq \beta \leq \theta/4$, it is maximized when β equals 0. Hence $\boldsymbol{c}$ needs to be at least $\cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4))$.

If $\delta(v_{j-1}, v_j) \leq |v_{j-1}d| + |dv_j| \cdot \boldsymbol{c}$, we note that $\gamma = \theta/4 + \beta$. Hence Lemma 3 gives that the inequality holds when $\boldsymbol{c} \geq (\cos(\beta - \theta/4) - \sin\beta)/(\cos(\theta/2 - \beta) - \sin(\theta/4 + \beta))$, which is equal to $\cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4))$. □

When looking at two vertices u and w in the constrained $\theta_{(4k+3)}$-graph and $\theta_{(4k+5)}$-graph, we notice that when the angle between uw and the bisector of T_{uw} is α, the angle between wu and the bisector of T_{wu} is $\theta/2 - \alpha$. Hence the worst case spanning ratio becomes the minimum of the spanning ratio when looking at T_{uw} and the spanning ratio when looking at T_{wu}.

Theorem 6. *The constrained $\theta_{(4k+3)}$-graph and $\theta_{(4k+5)}$-graph are $\frac{\cos\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{3\theta}{4}\right)}$-spanners of Vis$(P, S)$.*

Proof. The spanning ratio of the constrained $\theta_{(4k+3)}$-graph and $\theta_{(4k+5)}$-graph is at most:

$$\min \left\{ \begin{array}{l} \frac{\cos\alpha}{\cos\left(\frac{\theta}{2}\right)} + \frac{\left(\cos\alpha \cdot \tan\left(\frac{\theta}{2}\right) + \sin\alpha\right) \cdot \cos\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{3\theta}{4}\right)}, \\ \frac{\cos\left(\frac{\theta}{2} - \alpha\right)}{\cos\left(\frac{\theta}{2}\right)} + \frac{\left(\cos\left(\frac{\theta}{2} - \alpha\right) \cdot \tan\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2} - \alpha\right)\right) \cdot \cos\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{3\theta}{4}\right)} \end{array} \right\}$$

Since $\cos\alpha/\cos(\theta/2) + (\cos\alpha \cdot \tan(\theta/2) + \sin\alpha) \cdot \boldsymbol{c}$ is increasing for $\alpha \in [0, \theta/2]$, for $\theta \leq 2\pi/7$, the minimum of these two functions is maximized when the two functions are equal, i.e. when $\alpha = \theta/4$. Thus the constrained $\theta_{(4k+3)}$-graph and

$\theta_{(4k+5)}$-graph has spanning ratio at most:

$$\frac{\cos\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right)} + \frac{\left(\cos\left(\frac{\theta}{4}\right)\cdot\tan\left(\frac{\theta}{2}\right)+\sin\left(\frac{\theta}{4}\right)\right)\cdot\cos\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{3\theta}{4}\right)} = \frac{\cos\left(\frac{\theta}{4}\right)\cdot\cos\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)\cdot\left(\cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{3\theta}{4}\right)\right)}$$

□

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