

# On obstacle numbers

Vida Dujmović\*

Department of Computer Science and Electrical Engineering  
University of Ottawa  
Ottawa, Canada

vida.dujmovic@uottawa.ca

Pat Morin†

School of Computer Science  
Carleton University  
Ottawa, Canada

morin@scs.carleton.ca

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## Abstract

The obstacle number is a new graph parameter introduced by Alpert, Koch, and Laison (2010). Mukkamala et al. (2012) show that there exist graphs with  $n$  vertices having obstacle number in  $\Omega(n/\log n)$ . In this note, we up this lower bound to  $\Omega(n/(\log \log n)^2)$ . Our proof makes use of an upper bound of Mukkamala et al. on the number of graphs having obstacle number at most  $h$  in such a way that any subsequent improvements to their upper bound will improve our lower bound.

## 1 Introduction

The obstacle number is a new graph parameter introduced by Alpert, Koch, and Laison [2]. Let  $G = (V, E)$  be a graph, let  $\varphi : V \rightarrow \mathbb{R}^2$  be a one-to-one mapping of the vertices of  $G$  onto  $\mathbb{R}^2$  (hereafter called a *drawing* of  $G$ ), and let  $S$  be a set of connected subsets of  $\mathbb{R}^2$ . The pair  $(\varphi, S)$  is an *obstacle representation* of  $G$  when, for every pair of vertices  $u, w \in V$ , the edge  $uw$  is in  $E$  if and only if the closed line segment with endpoints  $\varphi(u)$  and  $\varphi(w)$  does not intersect any *obstacle* in  $S$ . An obstacle representation  $(\varphi, S)$  is an

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$h$ -obstacle representation if  $|S| = h$ . The *obstacle number* of a graph  $G$ , denoted by  $\text{obs}(G)$ , is the minimum value of  $h$  such that  $G$  has an  $h$ -obstacle representation.<sup>1</sup>

Note that obstacle representations of planar graphs using few obstacles often require drawings of those graphs that are far from crossing free. For example, any crossing-free drawing of the  $5 \times 5$  grid,  $G_{5 \times 5}$  shown in the left part of Figure 1 requires at least one obstacle in each of the sixteen internal faces (each of which has at least four sides).

It is somewhat surprising, therefore, that  $G_{5 \times 5}$  has obstacle number 1. The obstacle representation, illustrated on the right part of Figure 1 was given to us by Fabrizio Frati. In this figure, the single obstacle is drawn as a shaded region. Since at least one obstacle is clearly necessary to represent any graph other than a complete graph, this proves that  $\text{obs}(G_{5 \times 5}) = 1$ . (A similar drawing can be used to show that the  $a \times b$ , grid graph has obstacle number 1, for any integers  $a, b > 1$ .)

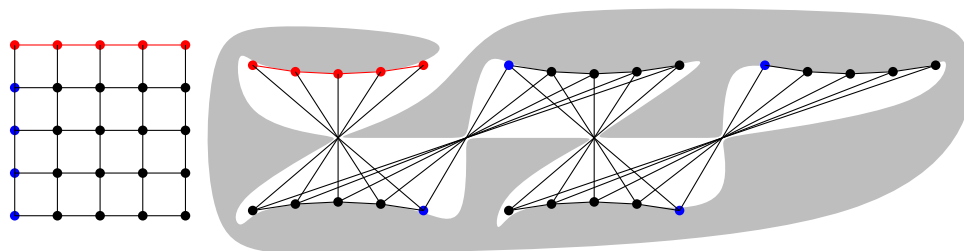


Figure 1: The  $5 \times 5$  grid graph has obstacle number 1.

Since their introduction, obstacle numbers have generated significant research interest [4, 5, 6, 7, 8, 9, 10]. A fundamental—and far from answered—question about obstacle numbers is that of determining the *worst-case obstacle number*,

$$\text{obs}(n) = \max\{\text{obs}(G) : G \text{ is a graph with } n \text{ vertices}\} ,$$

of a graph with  $n$  vertices.

For a graph  $G = (V, E)$ , we call elements of  $\binom{V}{2} \setminus E$  *non-edges* of  $G$ . The worst-case obstacle number  $\text{obs}(n)$  is obviously upper bounded by  $\binom{n}{2} \in O(n^2)$  since, by mapping the vertices of  $G$  onto a point set in sufficiently general position, one can place a small obstacle—even a single point—on the mid-point of each non-edge of  $G$ . No upper bound on  $\text{obs}(n)$  that is asymptotically better than  $O(n^2)$  is known.

More is known about lower bounds on  $\text{obs}(n)$ . Alpert, Koch, and Laison [2] initially show that the worst-case obstacle number is  $\Omega\left(\sqrt{\log n / \log \log n}\right)$  and posed as an open problem the question of determining if  $\text{obs}(n) \in \Omega(n)$ . Mukkamala et al. [7] showed that  $\text{obs}(n) \in \Omega(n / \log^2 n)$  and Mukkamala et al. [6] later increased this to  $\text{obs}(n) \in$

<sup>1</sup>Note that this definition of obstacle representation is more generous than that of Alpert, Koch, and Laison [2], which requires that the obstacles be polygonal and that the set of points determined by vertices of the obstacles and the image of  $\varphi$  not contain 3 collinear points. Since the current paper proves a lower bound on the obstacle number, this lower bound also applies to the original definition.

$\Omega(n/\log n)$ . In the current paper, we up the lower bound again by proving the following theorem:

**Theorem 1.** *For every integer  $n > 0$ ,  $\text{obs}(n) \in \Omega(n/(\log \log n)^2)$ , that is, there exists a sequence,  $\langle G_n \rangle_{n=1}^\infty$ , such that  $G_n$  is a graph with  $n$  vertices and such that  $\text{obs}(G) \in \Omega(n/(\log \log n)^2)$ .*

The proof of Theorem 1 makes use of an upper bound of Mukkamala et al. [6, Theorem 1] on the number of graphs having obstacle number at most  $h$  in such a way that any subsequent improvements on their upper bound will result in an improved lower bound on  $\text{obs}(n)$ .

Although some aspects of our proof are a little technical, the main idea is quite simple: Mukkamala et al. [6] show that, with probability at least  $1 - 2^{-\Omega(n^2)}$ , a random graph on  $n$  vertices has obstacle number at least  $\Omega(n/(\log n)^2)$ . Our proof trades off a lower probability for a higher obstacle number. When all is said and done, our proof shows that, with probability at least  $1 - 2^{-\Omega(n \log n)}$ , a random graph on  $n$  vertices has obstacle number at least  $\Omega(n/(\log \log n)^2)$ .

## 2 The Proof

Our proof strategy is an application of the probabilistic method [1]. We fix an arbitrary ordering,  $\pi$ , on the vertices of an Erdős–Rényi random graph,  $G = G_{n, \frac{1}{2}}$ . We then show that it is very unlikely that there is an obstacle representation,  $(\varphi, S)$  of  $G$  such that  $|S| \in o(n/(\log \log n)^2)$  and the lexicographic ordering of the points assigned to vertices by  $\varphi$  agrees with the ordering given by  $\pi$ . Here, “very unlikely” means that this occurs with probability  $p < 1/n!$ . Since there are only  $n!$  possible orderings of  $G$ 's vertices, we then apply the union bound which shows that with probability  $1 - pn! > 0$ , there is no obstacle representation of  $G$  using  $o(n/(\log \log n)^2)$  obstacles, that is,  $\text{obs}(G) \in \Omega(n/(\log \log n)^2)$ .

### 2.1 A Random Graph with a Fixed Ordering

We make use of the following theorem, due to Mukkamala, Pach, and Pálvölgyi [6, Theorem 1] about the number of  $n$ -vertex graphs with obstacle number at most  $h$ :

**Theorem 2** (Mukkamala, Pach, and Pálvölgyi 2012). *For any  $h \geq 1$ , the number of graphs having  $n$  vertices and obstacle number at most  $h$  is at most  $2^{O(hn \log^2 n)}$ .*

Recall that an Erdős–Rényi random graph  $G_{n, \frac{1}{2}}$  is a graph with  $n$  vertices and each pair of vertices is chosen as an edge or non-edge with equal probability and independently of every other pair of vertices [3]. The following lemma shows that, for random graphs, a fixed drawing is *very* unlikely to yield an obstacle representation with few obstacles. Recall that the *lexicographic ordering*,  $\prec$ , for points in the plane is defined as

$$(x_1, y_1) \prec (x_2, y_2) \text{ iff } x_1 < x_2 \text{ or } (x_1 = x_2 \text{ and } y_1 < y_2) .$$

**Lemma 1.** Let  $G = (V, E)$  be an Erdős–Rényi random graph  $G_{n, \frac{1}{2}}$ , let  $v_1, \dots, v_n$  be an ordering of the vertices in  $V$  that is independent of the choices of edges in  $G$ , and let  $(\varphi, S)$  be an obstacle representation of  $G$  using the minimum number of obstacles subject to the constraint that

$$\varphi(v_1) \prec \varphi(v_2) \prec \dots \varphi(v_n) ,$$

where  $\prec$  denotes the lexicographic ordering of points. Then, for any constant  $c > 0$ ,

$$\Pr\{|S| \in \Omega(n/(\log \log n)^2)\} \geq 1 - e^{-cn \log n} .$$

*Proof.* Fix some integer  $k = k(n) \in \omega_n(1)$  to be specified later and first consider the subgraph  $G_0$  of  $G$  induced by the vertices  $v_1, \dots, v_k$ . Applying Theorem 2 with  $n = k$  and  $h = \alpha k / \log^2 k$ , we obtain

$$\Pr\{\text{obs}(G_0) \leq \alpha k / \log^2 k\} \leq \frac{2^{O(\alpha k^2)}}{2^{\binom{k}{2}}} \leq e^{-\beta k^2} , \quad (1)$$

where  $\beta > 0$  for a sufficiently small constant  $\alpha > 0$ , and sufficiently large  $k$ . Note that, if  $\text{obs}(G_0) \geq h$ , then, in the obstacle representation  $(\varphi, S)$ , there must be at least  $h - 1$  obstacles of  $S$  that are contained in the convex hull of  $\varphi(v_1), \dots, \varphi(v_k)$ ; this is because the obstacle representation  $(\varphi, S)$  can be turned into an obstacle representation of  $G_0$  by merging all obstacles that are not contained in the convex hull of  $\varphi(v_1), \dots, \varphi(v_k)$ .

Let  $m = \lfloor n/k \rfloor$  and notice that the preceding argument applies to any subset  $V_i = \{v_{ki+1}, \dots, v_{(k+1)i}\}$  of vertices, for any  $i \in \{0, \dots, m - 1\}$ . That is, Equation (1) shows that, with probability at least  $1 - 2^{-\Omega(k^2)}$ , the obstacle number of the subgraph  $G_i$  induced by  $V_i$  is  $\Omega(k / \log^2 k)$ . If this occurs, then  $S$  has  $\Omega(k / \log^2 k)$  obstacles that are completely contained in the convex hull of  $V_i$ . In particular, the obstacles contained in the convex hull of  $V_i$  are different from the obstacles contained in the convex hull of  $V_j$ , for all  $j \neq i$ .

We are proving a lower bound on the number of obstacles, so we are worried about the case where the number of convex hulls that do *not* contain at least  $\alpha k / \log^2 k$  obstacles exceeds  $m/e$ .<sup>2</sup> The number of convex hulls,  $M$ , not containing at least  $\alpha k / \log^2 k$  obstacles is dominated by a binomial( $m, e^{-\beta k^2}$ ) random variable. Using Chernoff's bound on the tail of a binomial random variable,<sup>3</sup> we have that

$$\begin{aligned} \Pr\{M \geq m/e\} &= \Pr\{M \geq (1 + \delta)\mu\} && \text{(where } \mu = me^{-\beta k^2} \text{ and } \delta = e^{\beta k^2 - 1} - 1) \\ &\leq \left( \frac{e^\delta}{(1 + \delta)^{1 + \delta}} \right)^\mu \\ &\leq \left( \frac{e^{e\beta k^2}}{(e^{\beta k^2 - 1} - 1)^{e^{\beta k^2 - 1}}} \right)^{me^{-\beta k^2}} \end{aligned}$$

<sup>2</sup>Euler's constant  $e = \lim_{n \rightarrow \infty} (1 + 1/n)^n$  is just a convenient constant to use here.

<sup>3</sup>Chernoff's Bound: For any binomial( $m, p$ ) random variable,  $B$ , any  $\delta > 0$  and  $\mu = mp$ ,

$$\Pr\{B \geq (1 + \delta)\mu\} \leq \left( \frac{e^\delta}{(1 + \delta)^{1 + \delta}} \right)^\mu .$$

$$\begin{aligned}
&= \left( \frac{e^{e^{\beta k^2}}}{e^{(\beta k^2 - 1)e^{\beta k^2 - 1}}} \right)^{me^{-\beta k^2}} \\
&= \frac{e^m}{e^{m(\beta k^2 - 1)e^{\beta k^2 - 1}e^{-\beta k^2}}} \\
&= \frac{e^m}{e^{m(\beta k^2 - 1)/e}} \\
&= e^{-\Omega(mk^2)} .
\end{aligned}$$

Taking  $k = c' \log n$ , for a sufficiently large constant,  $c'$ , and recalling that  $m = \lfloor n/k \rfloor$ , we obtain the desired result. In particular,

$$|S| \geq \Omega\left(\left(\frac{k}{\log^2 k}\right) \cdot (m - m/e)\right) = \Omega\left(n/(\log \log n)^2\right)$$

with probability at least

$$1 - e^{-\Omega(mk^2)} = 1 - e^{-\Omega(c'n \log n)} \geq 1 - e^{-cn \log n} ,$$

for all  $n$  greater than some sufficiently large constant  $n_0$ . For  $n \in \{1, \dots, n_0\}$ , the lemma is trivially satisfied since  $|S| \geq 0$  with probability  $1 \geq 1 - e^{-cn \log n}$ .  $\square$

## 2.2 Finishing Up

For completeness, we now spell out the proof of Theorem 1.

*Proof of Theorem 1.* Let  $G = (V, E)$  be an Erdős-Rényi random graph with  $n$  vertices with vertex set  $V = \{1, \dots, n\}$ . For every obstacle representation  $(\varphi, S)$  of  $G$ , there is an ordering on  $V$  given by the lexicographic ordering of the points  $\{\varphi(v) : v \in V\}$ .

By Lemma 1, the probability that a particular such ordering,  $v_1, \dots, v_n$ , allows an obstacle representation using  $o(n/(\log \log n)^2)$  obstacles is at most  $p \leq e^{-cn \log n}$  for every constant  $c > 0$ . In particular, for sufficiently large  $c$ , we have  $p < 1/n!$ . By the union bound the probability that there is any ordering that supports an obstacle representation of  $G$  with  $o(n/(\log \log n)^2)$  obstacles is at most

$$\hat{p} = p \cdot n! < 1 .$$

We deduce that there exists some graph,  $G'$ , with  $\text{obs}(G') \in \Omega(n/(\log \log n)^2)$ .  $\square$

## 3 Remarks

Our proof of Theorem 1 relates the problem of counting the number of  $n$ -vertex graphs with obstacle number at most  $h$  to the problem of determining the worst-case obstacle number of a graph with  $n$  vertices. Currently, we use Theorem 2 of Mukkamala et al. [7], which proves an upper bound of  $e^{O(hn \log^2 n)}$  on the number of  $n$ -vertex graphs with obstacle number at most  $h$ .

Any improvement on the upper bound for the counting problem will immediately translate into an improved lower bound on the worst-case obstacle number. Let  $f(h, k)$  denote the number of  $k$ -vertex graphs with obstacle number at most  $h$  and let

$$\hat{h}(k) = \max \left\{ h : f(h, k) \leq 2^{k^2/4} \right\} .$$

The quantity  $\hat{h}(k)$  is chosen so that a random graph on  $k$  vertices has probability at most  $2^{-\Omega(k^2)}$  of having obstacle number less than  $\hat{h}(k)$ ; Theorem 2 shows that  $\hat{h}(k) \in \Omega(k/(\log k)^2)$ . Our proof of Lemma 1 shows that there exist graphs with obstacle number at least  $\Omega(n\hat{h}(c \log n)/(c \log n))$ .

We note that our technique gives an improved lower bound until someone is able to prove that  $\hat{h}(n) \in \Omega(n)$ . At this point, our approach gives a lower bound worse than the trivial lower bound  $\hat{h}(n)$ .

We conjecture that improved upper bounds on  $f(h, n)$  that reduce the dependence on  $h$  are the way forward:

**Conjecture 1.**  $f(h, n) \leq 2^{g(n) \cdot o(h)}$ , where  $g(n) \in O(n \log^2 n)$ .

In support of this conjecture, we observe that an upper bound of the form  $f(1, n) \leq 2^{g(n)}$  is sufficient to give the crude upper bound  $f(h, n) \leq 2^{h \cdot g(n)}$  since any graph with an  $h$ -obstacle representation is the common intersection of  $h$  graphs that each have a 1-obstacle representation. That is, if  $\text{obs}(G) \leq h$ , then there exists  $E_1, \dots, E_h$  such that  $G = (V, \bigcap_{i=1}^h E_i)$  and  $\text{obs}(V, E_i) = 1$  for all  $i \in \{1, \dots, h\}$ . This seems like a very crude upper bound in which many graphs are counted multiple times. Conjecture 1 asserts that this argument overestimates the dependence on  $h$ .

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A previous draft of this article proved a version Lemma 1 for a fixed drawing,  $\varphi$ , and then went to great lengths to argue that the number of combinatorially distinct drawings was at most  $2^{O(n \log n)}$ . We are grateful to an anonymous referee who pointed out that the proof of Lemma 1 also holds when only the lexicographic ordering of the vertices is fixed, thereby eliminating the need to bound the number of combinatorially equivalent drawings.

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