An ϵ - Approximation Algorithm for Weighted Shortest Paths on Polyhedral Surfaces *

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Abstract. Let $\mathcal{P}$ be a simple polyhedron, possibly non-convex, whose boundary is composed of n triangular faces, and in which each face has an associated positive weight. The cost of travel through each face is the distance traveled multiplied by the face's weight. We present an ϵ-approximation algorithm for computing a weighted shortest path on $\mathcal{P}$, i.e. the ratio of the length of the computed path with respect to the length of an optimal path is bounded by $(1+\epsilon)$, for a given $\epsilon > 0$.We give a detailed analysis to determine the exact constants for the approximation factor. The running time of the algorithm is $O(mn \log mn + nm^2)$. The total number of Steiner points, m, added to obtain the approximation depends on various parameters of the given polyhedron such as the length of the longest edge, the minimum angle between any two adjacent edges of $\mathcal{P}$ and the minimum distance from any vertex to the boundary of the union of its incident faces and the ratio of the largest (finite) to the smallest face weights of $\mathcal{P}$. Lastly, we present an approximation algorithm with an improved running time of $O(mn \log mn)$, at the cost of trading off the constants in the path accuracy. Our results present an improvement in the dependency on the number of faces, n, to the recent results of Mata and Mitchell [10] by a multiplicative factor of $n^2/\log n$, and to that of Mitchell and Papadimitriou [11] by a factor of n^7.

1 Introduction

1.1 Problem Definition

Shortest path problems are among the fundamental problems studied in computational geometry and other areas such as graph algorithms, geographical information systems (GIS) and robotics.[5] Let s and t be two vertices on a given possibly non-convex polyhedron $\mathcal{P}$, in $\Re^3$, consisting of n triangular faces on its

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[5] We encountered several shortest path related problems in our R&D on GIS (see [15]); more specifically, e.g., in emergency response time modeling where emergency units are dispatched to emergency sites based on minimum travel times.

boundary, each face has an associated weight, denoted by a positive real number w_i. A Euclidean *shortest path* $\pi(s,t)$ between s and t is defined to be a path with minimum Euclidean length among all possible paths joining s and t that lie on the surface of $\mathcal{P}$. A *weighted shortest path* $\Pi(s,t)$ between s and t is defined to be a path with minimum cost among all possible paths joining s and t that lie on the surface of $\mathcal{P}$. The *cost* of the path is the sum of lengths of all segments, the path traverses in each face multiplied by the corresponding face weight. A path $\Pi'(s,t)$ between two points s and t is said to be an ϵ-approximation of a (true) shortest path $\Pi(s,t)$ between s and t, if $\frac{\Pi'(s,t)}{\Pi(s,t)} \leq 1+\epsilon$, for some $\epsilon > 0$. The problem addressed in this paper is to determine an ϵ-approximate shortest path between two vertices on a weighted polyhedron.

1.2 Related Work

Shortest path problems in computational geometry can be categorized by various factors which include the dimensionality of space, the type and number of objects or obstacles (e.g., polygonal obstacles, convex or non-convex polyhedra, ...), and the distance measure used (e.g., Euclidean, number of links, or weighted distances). Several research articles, including surveys (see [5, 13]), have been written presenting the state-of-the-art in this active field. Due to the lack of space, here we discuss those contributions which relate more directly to our work; these are in particular 3-dimensional weighted scenarios.

Mitchell and Papadimitriou [11] introduced the weighted region problem in which each face has an associated positive weight, denoted by a real number $w_i > 0$. They presented an algorithm that computes a path between two points in a weighted planar subdivision which is at most $(1+\epsilon)$ times the shortest weighted path cost. Their algorithm requires $O(n^8 L)$ time in the worst case, where $L = \log(nNW/w\epsilon)$ is a factor representing the bit complexity of the problem instance. Here N is the largest integer coordinate of any vertex of the triangulation and W (w) is the maximum (minimum) weight of any face of the triangulation. Johannson discussed a weighted distance model for injection molding [6]. Lanthier et al. [8] presented several practical algorithms for approximating shortest paths in weighted domains. In addition, to their experimental verification and time analysis, they provided theoretically derived bounds on the quality of approximation. More specifically, the cost of the approximation is no more than the shortest path cost plus an (additive) factor of $W|L|$, where L is the longest edge, W is the largest weight among all faces. They also used graph spanners to get at most β times the shortest path cost plus $\beta W|L|$, where $\beta > 1$ is an adjustable constant. Mata and Mitchell [10] presented an algorithm that constructs a graph (pathnet) which can be searched to obtain an approximate path; their path accuracy is $(1+\frac{W}{kw\theta_{min}})$, where θ_{min} is the minimum angle of any face of $\mathcal{P}$, W/w is the largest to smallest weight ratio and k is a constant that depends upon ϵ.

Table 1 compares the running times for the ϵ-approximation algorithms developed by [11], [10], and the one presented in this paper in the case where all vertices are given as integer coordinates. From the table we can clearly see that our algorithm improves substantially the dependence on n, but the dependence on the geometric parameters is somewhat worse. Since in many applications n is quite large (larger than 10^5) the objective for this work has been to find an ϵ-approximation algorithm, where the dependence on n is considerably smaller.

Algorithm	Running Time ($K = O(\frac{N^2 W}{\epsilon w})$)
Mitchell and Papadimitriou [11]	$O(n^8 \log(\frac{nK}{N}))$
Mata and Mitchell [10]	$O(n^3 K)$
Our Results	$O(n(K \log K) \log(nK \log K))$

Table 1. Comparison of weighted shortest path algorithms that rely on geometric precision parameters. N represents the largest integer coordinate.

Although the objective of [8] was different, the schemes are ϵ-approximations in which the dependence on n becomes comparable to [10] (see [9]).

1.3 Our Approach

Our approach to solving the problem is to discretize the polyhedron in a natural way, by placing Steiner points along the edges of the polyhedron (as in our earlier subdivision approach [8]). We construct a graph G containing the Steiner points as vertices and edges as those interconnections between Steiner points that correspond to segments which lie completely in the triangular faces of the polyhedron. The geometric shortest path problem on polyhedra is thus stated as a graph problem so that the existing efficient algorithms (and their implementations) for shortest paths in graphs can be used. One of the differences to [8] and to other somewhat related work (e.g., [8, 3, 7]) lies in the placement of Steiner points.

We introduce a logarithmic number of Steiner points along each edge of $\mathcal{P}$, and these points are placed in a geometric progression along an edge. They are chosen w.r.t. the vertex joining two edges of a face such that the distance between any two adjacent points on an edge is at most ϵ times the shortest possible path segment that can cross that face between those two points.

Our discretization method falls into the class of edge subdivision algorithms. Grid-based methods as introduced e.g., by Papadimitriou [12], are instances of this class. As concluded by Choi, Sellen and Yap [2]: *"... grids are a familiar practical technique in all of computational sciences. From a complexity theoretic viewpoint, such methods have been shunned in the past as trivial or uninteresting. This need not be so, as Papadimitriou's work has demonstrated. In fact, the grid methods may be the most practical recourse for solving some intractable problems. It would be interesting to derive some general theorems about these approaches"* Lanthier et al. [8] and Mata and Mitchell [10] are proofs of such practical methods based on edge subdivision.

A problem arises when placing these Steiner points near vertices of the face since the shortest possible segment becomes infinitesimal in length. A similar issue was encountered by Kenyon and Kenyon [7] and Das and Narasimhan [3] during their work on rectifiable curves on the plane and in 3-space, respectively. The problem arises since the distance between adjacent Steiner points, in the near vicinity of a vertex, would have to be infinitesimal requiring an infinite number of Steiner points. We address this problem by constructing spheres around the vertices which have a very small radius (at most ϵ times the shortest distance from the vertex to an edge that is not incident to the vertex). The graph construction procedure never adds Steiner points within these spheres centered around each vertex of the polyhedron. This allows us to put a lower bound on the length of the smallest possible edge that passes between two adjacent Steiner points and hence we are able to add a finite number of Steiner points. As a result, if the shortest path passes through one of these spheres, the approximate path may pass through the vertex of the polyhedra, corresponding to the center of the sphere.

We show that there exist paths in this graph with costs that are within $(1+\epsilon)$ times the shortest path costs. For the purpose of simplifying the proofs, we actually show that the approximation is within the bound of $(1+\frac{3-2\epsilon}{1-2\epsilon}\epsilon)$ times the shortest path length in the unweighted scenario and within the bound of $(1+(2+\frac{2W}{(1-2\epsilon)w})\epsilon)$ times the shortest cost in the weighted scenario where $0<\epsilon<\frac{1}{2}$ and $\frac{W}{w}$ is the largest to smallest weight ratio of the faces of $\mathcal{P}$. The desired ϵ-approximation is achieved by dividing ϵ by $\frac{3-2\epsilon}{1-2\epsilon}$ or $(2+\frac{2W}{(1-2\epsilon)w})$ for the unweighted and weighted case, respectively. We can simplify the bounds of our algorithm when $\epsilon<1/6$. The bounds become $(1+4\epsilon)$ and $(1+(2+3\frac{W}{w})\epsilon)$ for the unweighted and weighted case, respectively. The running time of our algorithm is the cost for computing the graph G plus that of running a shortest path algorithm in G. The graph consists of $|V|=nm$ vertices and $|E|=nm^2$ edges where $m=O(\log_\delta(|L|/r))$, $|L|$ is the length of the longest edge, r is ϵ times the minimum distance from any vertex to the boundary of the union of its incident faces (denoted as minimum height h of any face), and $\delta\geq 1+\epsilon\sin\theta$, where θ is the minimum angle between any two adjacent edges of $\mathcal{P}$.

We also provide an algorithm to compute a subgraph G^* of G, on the same vertex set as that of G, but with only $O(nm\log nm)$ edges. G^* has the property that for any edge $\overline{uv}\in G$, there exists a path in G^* whose length is at most $(1+\epsilon)\|\overline{uv}\|$. This results in an ϵ-approximation algorithm for the shortest path problem and it runs in $O(nm\log nm)$ time. (To study the entries in Table 1, set $h=\Omega(1/N)$ and $\sin\theta=\Omega(1/N^2)$, where the vertices are assumed to have integer coordinates bounded by N).

Our analysis reveals the exact relationship between the geometric parameters and the algorithm's running time. The dependence on geometric parameters is an interesting feature of several approximation geometric algorithms. Many researchers have advocated the use of geometric parameters in analyzing the performance of geometric algorithms, and our result indicates that if the geometric parameters are "well-behaved" then the asymptotic complexity of our algorithms is several orders of magnitude better than existing ones. One of the conclusions from our study is that while studying the performance of geometric algorithms, geometric parameters (e.g. fatness, density, aspect ratio, longest, closest) should not be ignored, and in fact it could potentially be very useful to express the performance that includes the relevant geometric parameters.

2 Preliminaries

Let s and t be two vertices of a triangulated polyhedral surface $\mathcal{P}$ with n faces. A weight $w_i > 0$ is associated with each face $f_i \in \mathcal{P}$ such that the cost of travel through f_i is the distance traveled times w_i. Define W and w to be the maximum and minimum weight of all $w_i, 1 \leq i \leq n$, respectively.

Property 1. An edge of $\mathcal{P}$ cannot have a weight greater than its adjacent faces.

Let L be the longest edge of $\mathcal{P}$. Let $\pi(s,t)$ be a shortest Euclidean length path between s and t that remains on $\mathcal{P}$ with path length $|\pi(s,t)|$. Let $s_1, s_2, ..., s_k$ be the segments of $\pi(s,t)$ passing through faces $f_1, f_2, ..., f_k$. Similarly, define $\Pi(s,t)$ to be a shortest weighted cost path between s and t that remains on $\mathcal{P}$ with weighted cost denoted as $\|\Pi(s,t)\|$. Define $G(V,E) = G_1 \cup G_2 \cup ... G_n$ to be a graph such that $G_i, 1 \leq i \leq n$ is a subgraph created on face f_i with vertices lying on edges of f_i and edges of G_i lying across face f_i. Let $E(G_i)$ represent the edges of G_i and $V(G_i)$ represent the vertices of G_i. Let $\pi'(s,t) = s'_1, s'_2, ..., s'_k$ be a path in G passing through the same faces as $\pi(s,t)$ with length $|\pi'(s,t)|$. Similarly let $\Pi'(s,t)$ be a path in G passing through the same faces as $\Pi(s,t)$ with weighted cost $\|\Pi'(s,t)\|$.

Let v be a vertex of $\mathcal{P}$. Define h_v to be the minimum distance from v to the boundary of the union of its incident faces. Define a polygonal cap C_v, called a *sphere*, around v, as follows. Let $r_v = \epsilon h_v$ for some $0 < \epsilon$. Let r be the minimum r_v over all v. Let vuw be a triangulated face incident to v. Let u' (w') be at the distance of r_v from v on vu (vw). This defines a triangular sub-face $vu'w'$ of vuw. The sphere C_v around v consists of all such sub-faces incident at v.

Property 2. The distance between any two spheres C_{v_a} and C_{v_b} is greater than $(1-2\epsilon)h_{v_a}$.

Define θ_v to be the minimum angle (measured in 3D) between any two edges of $\mathcal{P}$ that are incident to v. Let θ be the minimum θ_v. A weighted path may *critically use* an edge of $\mathcal{P}$ by traveling along it and then reflecting back into the face [11]. We distinguish between two types of path segments of a shortest path: 1) *face-crossing* segments which cross a face and do not critically use an edge, and 2) *edge-using* segments which lie along an edge (critically using it). In the unweighted domain, edge-using segments span the entire length of an edge in $\mathcal{P}$ and a face can only be crossed once by a shortest path [14]. However, in the weighted domain, a face may be crossed more than once and so a weighted shortest path may have $\theta(n^2)$ segments, see [11].

3 An ϵ - Approximation Scheme

This section presents the approximation scheme by first describing the computation of the graph G, which discretize the problem. The construction of G

depends on the choice of ϵ, so we assume that a positive $\epsilon < \frac{1}{2}$ has been chosen and is fixed. A shortest path in G between s and t will be the ϵ-approximation path $\pi'(s,t)$ (or $\Pi(s,t)$) that we report.

Our proof strategy is as follows. We present a construction to show that there exists a path, $\pi'(s,t)$ (respectively, $\Pi'(s,t)$), between s and t in G, with cost at most $(1+\frac{3-2\epsilon}{1-2\epsilon}\epsilon)$ (respectively, $(1+(2+\frac{2W}{(1-2\epsilon)w})\epsilon)$) times the cost of $\pi(s,t)$ (respectively, $\Pi(s,t)$). Consider a shortest path $\Pi(s,t)$ in $\mathcal{P}$. It is composed of several segments which go through faces, and/or along edges, and/or through spheres around a vertex. For segments of $\Pi(s,t)$ that are not completely contained inside spheres, we show that there exists an appropriate edge in the graph. For segments that are lying inside a sphere, we use a "charging" argument. (Due to the lack of space, proofs are omitted; see [9] for details.)

3.1 An Algorithm to Compute the Graph

For each vertex v of face f_i we do the following: Let e_q and e_p be the edges of f_i incident to v. First, place Steiner points on edges e_q and e_p at distance r_v from v; call them q_1 and p_1, respectively. By definition, $|\overline{vq_1}| = |\overline{vp_1}| = r_v$. Define $\delta = (1+\epsilon\cdot\sin\theta_v)$ if $\theta_v < \frac{\pi}{2}$, otherwise $\delta = (1+\epsilon)$. We now add Steiner points $q_2, q_3, ..., q_{\mu_q-1}$ along e_q such that $|\overline{vq_j}| = r_v\delta^{j-1}$ where $\mu_q = \log_\delta(|e_q|/r_v)$. Similarly, add Steiner points $p_2, p_3, ..., p_{\mu_p-1}$ along e_p, where $\mu_p = \log_\delta(|e_p|/r_v)$. Define $\mathrm{dist}(a,e)$ as the minimum distance from a point a to an edge e. The segment from a to e will be a perpendicular to e. This strategy creates sets of Steiner points along edges e_q and e_p (see Figure 1a).

Claim 3.11. *$|\overline{q_iq_{i+1}}| = \epsilon\cdot dist(q_i,e_p)$ and $|\overline{p_jp_{j+1}}| = \epsilon\cdot dist(p_j,e_q)$ where $0 < i < \mu_q$ and $0 < j < \mu_p$.*

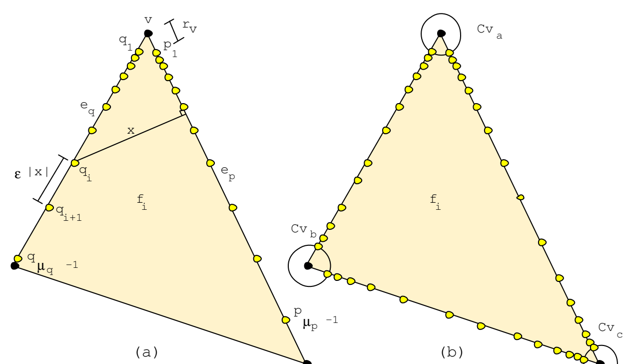

Fig. 1. a) Placement of Steiner points on the edges of f_i that are incident to vertex v. b) Results of merging Steiner points along edges.

Since we have added Steiner points based on the minimum angle θ_v about v, we obtain "concentric parallel wavefronts" centered at v consisting of Steiner point layers along the incident edges of v. Since this construction is made for each vertex of a face f_i, there will be two overlapping sets of Steiner points on each edge of f_i. To eliminate this overlap, we reduce the number of Steiner points on each edge. If two sets of Steiner points on an edge originate from the endpoints of an edge e, we determine the point on e where the interval sizes from each set are equal and eliminate all larger intervals. Intuitively, intervals are eliminated from one set if there are small intervals in the other set that overlap with it (see Figure 1b). The vertices of G_i will be Steiner points as well as the vertices of $\mathcal{P}$ defining f_i. The edges of G_i form a complete graph on its vertices. The graph G is defined to be the union $G_1 \cup G_2 \cup ... G_n$.

Claim 3.12. *G is connected.*

Claim 3.13. *At most $m \leq 2(1 + \log_\delta(|L|/r))$ Steiner points are added to each edge of f_i, for $1 \leq i \leq n$.*

Claim 3.14. *G has $O(n \log_\delta(|L|/r))$ vertices and $O(n(\log_\delta(|L|/r))^2)$ edges.*

Theorem 1. *Let $0 < \epsilon < \frac{1}{2}$. Let $\mathcal{P}$ be a simple polyhedron with n faces and let s and t be two of its vertices. An approximation $\pi'(s,t)$ of a Euclidean shortest path $\pi(s,t)$ between s and t can be computed such that $|\pi'(s,t)| \leq (1 + \frac{3-2\epsilon}{1-2\epsilon}\epsilon)|\pi(s,t)|$. An approximation $\Pi'(s,t)$ of a weighted shortest path $\Pi(s,t)$ between s and t can be computed such that $\|\Pi'(s,t)\| \leq (1 + (2 + \frac{2W}{(1-2\epsilon)w})\epsilon)\|\Pi(s,t)\|$. The approximations can be computed in $O(mn \log mn + nm^2)$ time where $m = \log_\delta \frac{|L|}{r}$, and $\delta = (1 + \epsilon \sin\theta)$.*

Proof. For both cases, we show that there exists a path in G that satisfies the claimed bounds using Lemma 2 and Lemma 4, respectively. Dijkstra's algorithm will either compute this path or a path with equal or better cost, and therefore the path computed by Dijkstra's algorithm as well satisfies the claimed approximation bounds. The running time of the algorithm follows from the size of the graph as stated in Claim 3.14. A variant of Dijkstra's algorithm using Fibonacci heaps [4] is employed to compute the path in the stated time bounds. □

3.2 Proof of Correctness

Consider a subgraph G_j, $1 \leq j \leq n$, as defined above. Let v be a vertex of a face f_j with edges e_p and e_q incident to v. We need the following technical lemma.

Lemma 1. *Let s_i be the smallest segment contained within f_j such that one endpoint of s_i intersects e_q between q_i and q_{i+1} and the other endpoint intersects e_p. It holds that $|\overline{q_i q_{i+1}}| \leq \epsilon|s_i|$. Furthermore, if $\theta_v < \frac{\pi}{2}$ then s_i is a perpendicular bisector to e_p and if $\theta_v \geq \frac{\pi}{2}$ then $|s_i| \geq |\overline{v q_i}|$.*

Let s_i be a segment of $\pi(s,t)$ (or $\Pi(s,t)$) crossing face f_i. Each s_i, must be of one of the following types:

i) $s_i \cap C_v = \emptyset$, ii) $s_i \cap C_v =$ subsegment of s_i, or iii) $s_i \cap C_v = s_i$.

Let $C_{\sigma_1}, C_{\sigma_2}, ..., C_{\sigma_\kappa}$ be a sequence of spheres (listed in order from s to t) intersected by type ii) segments of $\pi(s,t)$ such that $C_{\sigma_j} \neq C_{\sigma_{j+1}}$. Now define subpaths of $\pi(s,t)$ (and $\Pi(s,t)$) as being one of two kinds:

Definition 1. Between-sphere subpath: *A path consisting of a type ii) segment followed by zero or more consecutive type i) segments followed by a type ii) segment. These subpaths will be denoted as $\pi(\sigma_j, \sigma_{j+1})$ ($\Pi(\sigma_j, \sigma_{j+1})$ for weighted case) whose first and last segments intersect C_{σ_j} and $C_{\sigma_{j+1}}$, respectively. We will also consider paths that begin or/and end at a vertex to be a degenerate case of this type of path containing only type i) segments.*

Definition 2. Inside-sphere subpath: *A path consisting of one or more consecutive type iii) segments all lying within the same C_{σ_j}; these are denoted as $\pi(\sigma_j)$ ($\Pi(\sigma_j)$ for weighted case).*

Note that inside-sphere subpaths of $\pi(s,t)$ (and $\Pi(s,t)$) always lie between two between-sphere subpaths. That is, $\pi(\sigma_j)$ lies between $\pi(\sigma_{j-1}, \sigma_j)$ and $\pi(\sigma_j, \sigma_{j+1})$.

Claim 3.21. *Let s_i be a type i) segment with one endpoint between Steiner points q_j and q_{j+1} on edge e_q of a face f_i and the other endpoint between Steiner points p_k and p_{k+1} on edge e_p of f_i.*
Then $\max(\min(|\overline{q_j p_k}|, |\overline{q_j p_{k+1}}|), \min(|\overline{q_{j+1} p_k}|, |\overline{q_{j+1} p_{k+1}}|)) \leq (1+\epsilon)|s_i|$.

Claim 3.22. *Let s_i be a type ii) segment crossing edge e_q of f_i between Steiner points q_j and q_{j+1} and crossing e_p between v and Steiner point p_1, where $j \geq 1$ and v is the vertex common to e_q and e_p. Then $|\overline{q_1 q_j}|$ and $|\overline{q_1 q_{j+1}}|$ are less than $(1+\epsilon)|s_i|$.*

Bounding the Unweighted Approximation: We first describe the construction of an approximate path $\pi'(s,t)$ in G, given a shortest path $\pi(s,t)$ in $\mathcal{P}$. Consider a between-sphere subpath $\pi(\sigma_j, \sigma_{j+1})$, which consists of type i) and type ii) segments only. First examine a type i) segment s_i of $\pi(\sigma_j, \sigma_{j+1})$ that passes through edges e_q and e_p of face f_i. Assume s_i intersects e_q between Steiner points q_j and q_{j+1} and also intersects e_p between Steiner points p_k and p_{k+1}, where $j, k \geq 1$. The approximated path is chosen such that it enters face f_i through Steiner point q_j or q_{j+1}. W.l.o.g. assume that the approximated path enters f_i at q_j. Choose s'_i to be the shortest of $\overline{q_j p_k}$ and $\overline{q_j p_{k+1}}$. It is easily seen that $\pi'(\sigma_j, \sigma_{j+1})$ is connected since adjacent segments s'_{i-1} and s'_{i+1} share an endpoint (i.e., a Steiner point).

Now examine a type ii) segment of $\pi(\sigma_j, \sigma_{j+1})$; this can appear as the first or last segment. W.l.o.g. assume that it is the first segment. Let this segment enter f_i between Steiner points q_j and q_{j+1} and exit between vertex v_{σ_j} and Steiner point p_1 on e_p. Let $s'_i = \overline{q_1 q_j}$ (if s_i is the last segment, then we either choose s'_i to be $\overline{q_1 q_j}$ or $\overline{q_1 q_{j+1}}$ depending on at which Steiner point the approximated path up to f_i enters f_i). It is easily seen that the combination of these approximated segments forms a connected chain of edges in G which we will call $\pi'(\sigma_j, \sigma_{j+1})$. One crucial property of $\pi'(\sigma_j, \sigma_{j+1})$ is that it begins at a point where C_{σ_j} intersects an edge of $\mathcal{P}$ and ends at a point where $C_{\sigma_{j+1}}$ intersects an edge of $\mathcal{P}$.

Consider two consecutive between-sphere subpaths of $\pi(s,t)$, say $\pi'(\sigma_{j-1}, \sigma_j)$ and $\pi'(\sigma_j, \sigma_{j+1})$. They are disjoint from one another, however, the first path ends at sphere C_{σ_j} and the second path starts at C_{σ_j}. Join the end of $\pi'(\sigma_{j-1}, \sigma_j)$ and the start of $\pi'(\sigma_j, \sigma_{j+1})$ to vertex v_{σ_j} by two segments (which are edges of G). These two segments together will be denoted as $\pi'(\sigma_j)$. This step is repeated for each consecutive pair of between-sphere subpaths so that all subpaths are joined to form $\pi'(s,t)$. (The example of Figure 2 shows how between-sphere subpaths are connected to inside-sphere subpaths.) Constructing a path in this manner results in a continuous path that lies on the surface of $\mathcal{P}$.

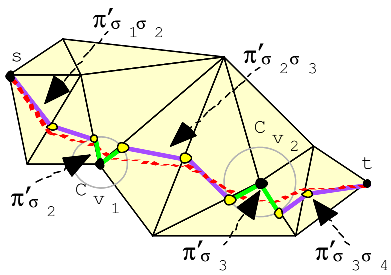

Fig. 2. An example showing the between-sphere and inside-sphere subpaths that connect to form the approximated path $\pi(s,t)$.

Claim 3.23. *Let $\pi'(\sigma_{j-1}, \sigma_j)$ be a between-sphere subpath of $\pi'(s,t)$ corresponding to an approximation of $\pi(\sigma_{j-1}, \sigma_j)$ where $1 < j \le \kappa$. Then $|\pi'(\sigma_j)| \le \frac{2\epsilon}{1-2\epsilon}|\pi(\sigma_{j-1}, \sigma_j)|$, where $0 < \epsilon < \frac{1}{2}$.*

Proof. From Property 2, the distance between $C_{\sigma_{j-1}}$ and C_{σ_j} must be at least $(1-2\epsilon)h_{v_{\sigma_j}}$. Since $\pi(\sigma_{j-1}, \sigma_j)$ is a between-sphere subpath, it intersects both $C_{\sigma_{j-1}}$ and C_{σ_j}. Thus $|\pi(\sigma_{j-1}, \sigma_j)| \ge (1-2\epsilon)h_{v_{\sigma_j}}$. By definition, $\pi'(\sigma_j)$ consists of exactly two segments which together have length satisfying $|\pi'(\sigma_j)| = 2r_{v_{\sigma_j}} = 2\epsilon h_{v_{\sigma_j}}$. Thus, $|\pi(\sigma_{j-1}, \sigma_j)| \ge \left(\frac{1-2\epsilon}{2\epsilon}\right)|\pi'(\sigma_j)|$ which can be re-written as $|\pi'(\sigma_j)| \le \frac{2\epsilon}{1-2\epsilon}|\pi(\sigma_{j-1}, \sigma_j)|$. □

Lemma 2. *If $\pi(s,t)$ is a shortest path in $\mathcal{P}$, where s and t are vertices of $\mathcal{P}$ then there exists an approximated path $\pi'(s,t) \in G$ for which $|\pi'(s,t)| \le \left(1 + \frac{3-2\epsilon}{1-2\epsilon}\epsilon\right)|\pi(s,t)|$, where $0 < \epsilon < \frac{1}{2}$.*

Proof. Using the results of Claim 3.22 and Claim 3.23, we can "charge" the cost of each inside-sphere subpath $\pi'(\sigma_j)$ to the between-sphere subpath $\pi'(\sigma_{j-1}, \sigma_j)$ as follows: $|\pi'(\sigma_{j-1}, \sigma_j)| + |\pi'(\sigma_j)| \le (1+\epsilon)|\pi(\sigma_{j-1}, \sigma_j)| + \left(\frac{2\epsilon}{1-2\epsilon}\right)|\pi(\sigma_{j-1}, \sigma_j)| = \left(1 + \frac{3-2\epsilon}{1-2\epsilon}\epsilon\right)|\pi(\sigma_{j-1}, \sigma_j)|$. The union of all subpaths $\pi'(\sigma_{j-1}, \sigma_j)$ and $\pi'(\sigma_j)$ form $\pi'(s,t)$ where $2 \le j \le \kappa$. Hence, we have bounded $|\pi'(s,t)|$ w.r.t. the between-sphere subpaths of $\pi(s,t)$. Therefore
$|\pi'(s,t)| \le \left(1 + \frac{3-2\epsilon}{1-2\epsilon}\epsilon\right)\sum_{j=2}^{\kappa}|\pi(\sigma_{j-1}, \sigma_j)| \le \left(1 + \frac{3-2\epsilon}{1-2\epsilon}\epsilon\right)|\pi(s,t)|$. □

Bounding the Weighted Approximation: Given a shortest path $\Pi(s,t)$, we construct a path $\Pi'(s,t)$ in a similar manner as in the unweighted scenario. However, we must consider the approximation of edge-using segments since they may no longer span the full length of the edge which they are using. Consider an edge-using segment s_i of $\Pi(s,t)$ on edge e_p of $\mathcal{P}$ with endpoints lying in Steiner point intervals $[p_y, p_{y+1}]$ and $[p_{u-1}, p_u]$ along e_p, where $y < u$. Let s_{i-1} and s_{i+1}, respectively, be the two crossing segments representing the predecessor and successor of s_i in the sequence of segments in $\pi(s,t)$. We will assume that two such edges exist although it is possible that s_{i-1} and s_i meet at a vertex of $\mathcal{P}$; which can easily be handled as well. We choose an approximation s'_i of s_i to be one of $\overline{p_y p_{u-1}}$ or $\overline{p_{y+1} p_{u-1}}$ depending on whether s'_{i-1} intersects e_p at p_y or p_{y+1}, respectively. Note that we make sure to choose s'_i so that it is connected to s'_{i-1}. Of course, s'_{i+1} will also be chosen to ensure connectivity with s'_i. In the degenerate case where $u = y+1$, then there is no approximation for s_i. Instead, s'_{i-1} is connected directly to s'_{i+1}. In fact, Dijkstra's algorithm will never choose such a subpath since it does not make use of e_p. However, the path it does choose will be better than this, so our bound will hold for this better path as well.

Claim 3.24. *Let s_i be an edge-using segment of $\Pi(\sigma_j, \sigma_{j+1})$ and let s_{i-1} be the segment of $\Pi(\sigma_j, \sigma_{j+1})$ preceding s_i. There exists a segment s'_i of $\Pi'(\sigma_j, \sigma_{j+1})$ for which $\|s'_i\| \leq \|s_i\| + \epsilon\|s_{i-1}\|$.*

Lemma 3. *If $\Pi'(\sigma_{j-1}, \sigma_j)$ is a between-sphere subpath of $\Pi'(s,t)$ corresponding to an approximation of $\Pi(\sigma_{j-1}, \sigma_j)$ then $\|\Pi'(\sigma_{j-1}, \sigma_j)\| \leq (1+2\epsilon)\|\Pi(\sigma_{j-1}, \sigma_j)\|$.*

Claim 3.25. *Let $\Pi'(\sigma_{j-1}, \sigma_j)$ be a between-sphere subpath of $\Pi'(s,t)$ corresponding to an approximation of $\Pi(\sigma_{j-1}, \sigma_j)$ then*
$\|\Pi'(\sigma_j)\| \leq \frac{2\epsilon W}{(1-2\epsilon)w}\|\Pi(\sigma_{j-1}, \sigma_j)\|$ *where* $0 < \epsilon < \frac{1}{2}$.

We have made the assumption that $\Pi'(\sigma_j)$ consists of segments passing through faces that have weight W. Although this may be true in the worst case, we could use the maximum weight of any face adjacent to v_{σ_j}, which typically would be smaller than W. In addition, we have assumed that $\Pi'(\sigma_{j-1}, \sigma_j)$ traveled through faces with minimum weight. We could determine the smallest weight of any face through which $\Pi'(\sigma_{j-1}, \sigma_j)$ passes and use that in place of w. This would lead to a better bound.

Lemma 4. *If $\Pi(s,t)$ is a shortest weighted path in $\mathcal{P}$, where s and t are vertices of $\mathcal{P}$ then there exists an approximated path $\Pi'(s,t) \in G$ such that $\|\Pi'(s,t)\| \leq (1 + (2 + \frac{2W}{(1-2\epsilon)w})\epsilon)\|\Pi(s,t)\|$ where $0 < \epsilon < \frac{1}{2}$.*

Proof. Using the results of Claim 3.25 and Lemma 3, it can be shown that
$\|\Pi'(\sigma_{j-1}, \sigma_j)\| + \|\Pi'(\sigma_j)\| \leq (1 + (2 + \frac{2W}{(1-2\epsilon)w})\epsilon)\|\Pi(\sigma_{j-1}, \sigma_j)\|$.
This essentially "charges" the length of an inside-sphere subpath to a between-sphere subpath. The union of all such subpaths form $\Pi'(s,t)$. This allows us to approximate $\Pi'(s,t)$ within the bound of $1 + (2 + \frac{2W}{(1-2\epsilon)w})\epsilon$ times the total cost of all the between-sphere subpaths of $\Pi(s,t)$. Since $\Pi(s,t)$ has cost at least that of its between-sphere subpaths, $\|\Pi'(s,t)\| \leq (1 + (2 + \frac{2W}{(1-2\epsilon)w})\epsilon)\|\Pi(s,t)\|$. ☐

4 Reduced Approximation Graph

We show that some of the edges of the approximation graph G can be removed, so that for the obtained graph G^*, our results hold with a slightly worse constant. Since the reduced graph has only $O(nm \log mn)$ edges the running time of the resulting algorithm will improve substantially.

The graph $G^* = (V(G), E(G^*))$ is a subgraph of G having the same vertex set and $E(G^*) \subset E(G)$. We describe the construction of G^* by describing the choice of edges for $E(G^*)$. All edges in $E(G)$ that are subsegments of the edges of $\mathcal{P}$ remain in $E(G^*)$. The vertices of $\mathcal{P}$ in G^* are adjacent only to their neighboring Steiner points. Now we consider a fixed Steiner point p and describe the edges of G^* incident to p. Assume that p lies on an edge e of $\mathcal{P}$. By our construction of G the point p is connected to all Steiner points that lie on one of the four (or less) edges of $\mathcal{P}$ sharing a face (triangle) with e. We describe the construction on one of these edges, say e_1, and let $q_1, \ldots q_k$ be the Steiner points on e_1. Let M be the point closest to p on interval (q_1, q_k). The edges $\overline{pq_1}$, and $\overline{pq_k}$ are in $E(G^*)$. We choose the edges joining p with points in the subintervals (q_1, M) and (M, q_k) as follows: Consider the interval (M, q_k) and define a sequence of points $x_0, x_1, \ldots, x_\kappa$ in this interval, so that $|\overline{x_{i-1}x_i}| = \varepsilon|\overline{px_{i-1}}|$. Observe that there is at least one Steiner point in each of the intervals (x_{i-1}, x_i), for $i = 1, \ldots, \kappa$. Now, for $i = 1, \ldots \kappa$, we denote by q^i the Steiner point closest to x_{i-1} in the interval (x_{i-1}, x_i) and define $\overline{pq^i}$ to be edges of G^*. By the same procedure we define the subset of the Steiner points in (M, q_1) and connect p to the points in this subset. Omitting the technical proof, we claim that the out-degree of any Steiner point is $O(\log m)$; hence G^* has $O(nm \log mn)$ edges and any edge e in G can be approximated by a path e^* in G^* so that $\|e^*\|$ is an ϵ- approximation of $\|e\|$. The result is summarized in the following theorem:

Theorem 2. *An ϵ-approximate weighted shortest path between two vertices on a polytope consisting of n triangular faces can be computed in $O(nm \log mn)$ time, where m is a constant that depends upon ϵ and the geometric parameters as discussed before.*

5 Conclusions and Ongoing Work

We have presented algorithms to compute ϵ-approximate paths on weighted polyhedra. More specifically, the algorithms compute paths from the source vertex s to all vertices, Steiner points which are introduced on edges of the polyhedron. The techniques described in this paper can be used to derive algorithms for shortest path queries, as discussed in [1]. An alternative approach, which we are investigating, is to compute the relevant portion of the subgraphs G_i on the fly. It is clear that in Dijkstra's algorithm when the current vertex v (with least cost) explores the edges incident to it, we don't have to explore all of them because of the following: suppose the approximate path to v is through an edge $\overline{uv}$, then from v we need to explore those edges which obey Snell's law with respect to $\overline{uv}$. We suspect that the total number of edges that needs to be explored with this modification will be substantially lower. Moreover, we do not have to sacrifice the accuracy of the path obtained.

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